

# Lagrangian stochastic models with specular boundary condition

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## Abstract

In this paper, we prove the well-posedness of a conditional McKean Lagrangian stochastic model endowing the specular boundary condition and the mean no-permeability condition in smooth bounded confinement domain  $\mathcal{D}$ . This result extends our previous work [5], where we dealt with the case where the confinement domain is the upper-half plane and where the specular boundary condition is introduced in generic Langevin process owing to some well known results on the law of the passage times at zero of the Brownian primitive. The extension to more general confinement domain exhibit more difficulties that can be handled by combining stochastic calculus and the analysis of kinetic equations. As a prerequisite for the nonlinear case, we construct a Langevin process confined in  $\overline{\mathcal{D}}$  and satisfying the specular boundary condition. We then use PDE techniques to construct the time-marginal densities of the nonlinear process from which we are able to exhibit the conditional McKean Lagrangian stochastic model.

**Key words:** Lagrangian stochastic model; No-mean permeability; Trace problems.

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## 1 Introduction

We are interested in the well-posedness of the stochastic process  $(X_t, U_t)_{0 \leq t \leq T}$ , for any arbitrary finite time  $T > 0$ , whose time-evolution is given by

$$\begin{cases} X_t = X_0 + \int_0^t U_s ds, \\ U_t = U_0 + \int_0^t B[X_s; \rho_s] ds + \sigma W_t + K_t, \\ K_t = - \sum_{0 < s \leq t} 2 (U_{s-} \cdot n_{\mathcal{D}}(X_s)) n_{\mathcal{D}}(X_s) \mathbb{1}_{\{X_s \in \partial \mathcal{D}\}}, \\ \rho_t \text{ is the probability density of } (X_t, U_t) \text{ for all } t \in (0, T], \end{cases} \quad (1.1)$$

where  $(W_t, t \geq 0)$  is a standard  $\mathbb{R}^d$ -Brownian motion, the diffusion  $\sigma$  is a positive constant and  $\mathcal{D}$  is a bounded domain of  $\mathbb{R}^d$ . Equation (1.1) provides a Lagrangian model describing at each time  $t$ , the position  $X_t$  and the velocity  $U_t$  of a particle confined within  $\overline{\mathcal{D}}$ .

The drift coefficient  $B$  is the mapping from  $\mathcal{D} \times L^1(\mathcal{D} \times \mathbb{R}^d)$  to  $\mathbb{R}^d$  defined by

$$B[x; \psi] = \begin{cases} \frac{\int_{\mathbb{R}^d} b(v) \psi(t, x, v) dv}{\int_{\mathbb{R}^d} \psi(t, x, v) dv} & \text{whenever } \int_{\mathbb{R}^d} \psi(t, x, v) dv \neq 0, \\ 0 & \text{otherwise} \end{cases} \quad (1.2)$$

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where  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  a given measurable function. Formally the function  $(t, x) \mapsto B[x; \rho_t]$  in (1.1) corresponds to the conditional expectation  $(t, x) \mapsto \mathbb{E}[b(U_t)/X_t = x]$  and the velocity equation in (1.1) rewrites

$$U_t = U_0 + \int_0^t \mathbb{E}[b(U_s)/X_s] ds + \sigma W_t + K_t.$$

The role of the càdlàg process  $K_t$  in the dynamics of  $U$  is to confine the component  $X$  in  $\overline{\mathcal{D}}$  by reflecting the velocity of the outgoing particle. This particular confinement is linked with the specular boundary condition:

$$\gamma(\rho)(t, x, u) = \gamma(\rho)(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), \quad dt \otimes d\sigma_{\partial\mathcal{D}} \otimes du\text{-a.e. on } (0, T) \times \partial\mathcal{D} \times \mathbb{R}^d, \quad (1.3)$$

where  $\sigma_{\partial\mathcal{D}}$  denotes the surface measure of  $\partial\mathcal{D}$  and where  $\gamma(\rho)$  stands for the trace of the probability density  $\rho$  on  $(0, T) \times \partial\mathcal{D} \times \mathbb{R}^d$ . As already noticed in [5, Corollary 2.4], under integrability and positiveness properties on  $\gamma(\rho)$ , the specular condition (1.3) implies the mean no-permeability boundary condition:

$$\frac{\int_{\mathbb{R}^d} (u \cdot n_{\mathcal{D}}(x)) \gamma(\rho)(t, x, u) du}{\int_{\mathbb{R}^d} \gamma(\rho)(t, x, u) du} = 0, \quad \text{for } dt \otimes d\sigma_{\partial\mathcal{D}}\text{-a.e. } (t, x) \in (0, T) \times \partial\mathcal{D}. \quad (1.4)$$

The function  $(t, x) \mapsto \frac{\int_{\mathbb{R}^d} (u \cdot n_{\mathcal{D}}(x)) \gamma(\rho)(t, x, u) du}{\int_{\mathbb{R}^d} \gamma(\rho)(t, x, u) du}$ , for  $x \in \partial\mathcal{D}$ , serves here as a formal representation of the normal component of the bulk velocity at the boundary  $\mathbb{E}((U_t \cdot n_{\mathcal{D}}(X_t))/X_t = x)$ , so that (1.4) can be seen as

$$\mathbb{E}((U_t \cdot n_{\mathcal{D}}(X_t))/X_t = x) = 0 \quad \text{for } dt \otimes d\sigma_{\partial\mathcal{D}}\text{-a.e. } (t, x) \in (0, T) \times \partial\mathcal{D}.$$

In view of (1.4), an appropriate notion of the trace of  $\rho$  is given with the following

**Definition 1.1.** *Let  $(\rho_t; t \in [0, T])$  be the time-marginal densities of a solution to (1.1). We say that  $\gamma(\rho) : \Sigma_T \rightarrow \mathbb{R}$  is the trace of  $(\rho_t; t \in [0, T])$  along  $\Sigma_T$  if the following properties hold.*

(i) *For  $dt \otimes d\sigma_{\partial\mathcal{D}}$  a.e.  $(t, x)$  in  $(0, T) \times \partial\mathcal{D}$ ,*

$$\int_{\mathbb{R}^d} |(v \cdot n_{\mathcal{D}}(x))| \gamma(\rho)(t, x, v) dv < +\infty, \quad (1.5a)$$

$$\int_{\mathbb{R}^d} \gamma(\rho)(t, x, v) dv > 0. \quad (1.5b)$$

(ii) *For all  $t$  in  $(0, T]$  and  $f$  in  $\mathcal{C}_c^\infty(\overline{Q_T})$ ,  $\gamma(\rho)$  satisfies the following Green formula:*

$$\begin{aligned} & \int_{\mathcal{D} \times \mathbb{R}^d} f(t, x, u) \rho_t(x, u) dx du - \int_{\mathcal{D} \times \mathbb{R}^d} f(0, x, u) \rho_0(x, u) dx du \\ & - \int_{Q_t} \left( \partial_s f(s, x, u) + \left( \frac{\sigma^2}{2} \triangle_u f + u \cdot \nabla_x f \right) (s, x, u) \right) \rho_s(x, u) ds dx du \\ & = - \int_{\Sigma_t} (u \cdot n_{\mathcal{D}}(x)) \gamma(\rho)(s, x, u) f(s, x, u) ds d\sigma_{\partial\mathcal{D}}(x) du. \end{aligned} \quad (1.6)$$

In addition to the well-posedness of (1.1), we prove that the solution admit a trace in a sense of Definition 1.1 and thus satisfy the specular and mean no-permeability boundary condition.

Our interest in the model (1.1) and its connection with (1.4) arises with the modeling of boundary conditions of the Lagrangian Stochastic Models for turbulent flows. These models are developed in the context of computational fluid mechanics and features a class of stochastic differential equations with singular coefficients (we refer to Bernardin *et al.* [2], Bossy *et al.* [6] for an account of the various theoretical and computational issues introduced by these models). The design of boundary conditions for the Lagrangian Stochastic Models according to some Dirichlet condition or some physical wall law, the analysis of their effects on the nonlinear dynamics and their momenta are among the current challenging issues raised by the use of Lagrangian Stochastic Models in Computational Fluid Dynamics.

In the kinetic theory of gases, the specular boundary condition belongs to the family of the Maxwell boundary conditions which model the interaction (diffusion and absorption phenomenon) between gas particles and solid surface (see Cercignani [10]). Specifically, the specular boundary condition models the particle reflection at the boundary on totally elastic wall (no loss of mass nor energy).

The intrinsic difficulty to the well-posedness of (1.1) lies in the study of the hitting times  $\{\tau_n, n \geq 0\}$  of the particle position  $(X_t)$  on the boundary  $\partial\mathcal{D}$ , defined by

$$\begin{cases} \tau_n = \inf\{t > \tau_{n-1} ; X_t \in \partial\mathcal{D}\} \text{ if } n \geq 1, \\ \tau_0 = 0 \end{cases}$$

and that must tend to infinity to ensure that  $(K_t)$  is well defined. By Girsanov theorem, it is not difficult to see that the sequence  $\{\tau_n, n \geq 0\}$  is related to the attaining times of the primitive of the Brownian motion on a smooth surface.

In the previous work [5], we established the well-posedness of (1.1) in the case where  $\mathcal{D}$  is the upper half-plane  $\mathbb{R}^{d-1} \times (0, +\infty)$ . In this situation, only one component of the process is confined in  $[0, +\infty)$ , then our construction of the confined process mainly relies on the results on the zero-sets of the primitive of one dimensional Brownian motion given in McKean [24] and Lachal [20]. To the best of our knowledge, estimates on these attaining times have only been considered in the case of bounded interval. Note also that in the case treated in [5], the trace problem (that is showing the existence of trace functions in the sense of Definition 1.1) had an explicit solution given by the explicit construction of the confined linear Langevin process.

Here some new difficulties are enhanced by the boundary reflection generalized to any smooth bounded domain  $\mathcal{D}$ . Those difficulties appear first in the construction of the confined linear Langevin process, second in the treatment of the McKean nonlinearity in (1.1) and in the verification of the mean-no-permeability condition.

The approach that we propose in this paper strongly mixes stochastic analysis with PDE analysis. One of the key point of our construction lie on obtaining some controls on this density of (1.1) and on resolving the trace problem for the density of the solution to (1.1).

## 1.1 Main result

The set of hypotheses for the main theorem are denoted  $(H)$ . In this set we distinguish  $(H_{\text{Langevin}})$  the hypotheses for the construction of the linear Langevin process, and  $(H_{\text{VFP}})$  the hypotheses for the well-posedness of the Vlasov-Fokker-Planck equation as follows:

$(H_{\text{Langevin}})$ -(i) The initial condition  $(X_0, U_0)$  are assumed to be distributed according to a given initial law  $\mu_0$  having its support in  $\mathcal{D} \times \mathbb{R}^d$  and such that  $\int_{\mathcal{D} \times \mathbb{R}^d} (|x|^2 + |u|^2) \mu_0(dx, du) < +\infty$ .

$(H_{\text{Langevin}})$ -(ii) The boundary  $\partial\mathcal{D}$  is a compact manifold of class  $C^3$ .

$(H_{\text{VFP}})$ -(i)  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a bounded measurable function.

$(H_{\text{VFP}})$ -(ii) The initial law  $\mu_0$  has a density  $\rho_0$  in the weighted  $L^2$  space  $L^2(\omega, \mathcal{D} \times \mathbb{R}^d)$  with  $\omega(u) := (1 + |u|^2)^{\frac{\alpha}{2}}$  for some  $\alpha > d$  (see the Notation section for a precise definition).

$(H_{\text{VFP}})$ -(iii) There exists two measurable functions  $\underline{P}_0, \overline{P}_0 : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  not identically equal to zero such that

$$\begin{aligned} 0 < \underline{P}_0(|u|) \leq \rho_0(x, u) \leq \overline{P}_0(|u|), \text{ a.e. on } \mathcal{D} \times \mathbb{R}^d; \\ \text{and } \int_{\mathbb{R}^d} (1 + |u|) \omega(u) |\overline{P}_0(|u|)|^2 du < +\infty. \end{aligned}$$

Let us precise the notion of solution that we consider. A probability measure  $\mathbb{Q}$  in the sample space  $\mathcal{T} := \mathcal{C}([0, T]; \overline{\mathcal{D}}) \times \mathbb{D}([0, T]; \mathbb{R}^d)$  is a solution in law to (1.1) if and only if, for all  $t \in [0, T]$ ,  $\mathbb{Q} \circ (x(t), u(t))^{-1}$  admits a density function  $\rho(t)$  with  $\rho(0) = \rho_0$  and there exists a  $\mathbb{R}^d$ -Brownian motion  $(w(t))$  under  $\mathbb{Q}$ ,

such that, for  $(x(t), u(t); t \in [0, T])$  the canonical process of  $\mathcal{T}$ ,  $\mathbb{Q}$ -a.s.,

$$\begin{cases} x(t) = x(0) + \int_0^t u(s) ds, \\ u(t) = u(0) + \int_0^t B[x(s); \rho(s)] ds + \sigma w(t) - \sum_{0 < s \leq t} 2(u(s^-) \cdot n_{\mathcal{D}}(x(s))) n_{\mathcal{D}}(x(s)) \mathbb{1}_{\{x(s) \in \partial \mathcal{D}\}}. \end{cases}$$

We further introduce the set

$$\Pi_{\omega} := \{ \mathbb{Q} \text{ probability measure on } \mathcal{T} \text{ s.t. for all } t \in [0, T], \mathbb{Q} \circ (x(t), u(t))^{-1} \in L^2(\omega; \mathcal{D} \times \mathbb{R}^d) \}.$$

**Theorem 1.2.** *Under (H), there exists a unique solution in law to (1.1) in  $\Pi_{\omega}$ .*

*Moreover the set of time-marginal densities  $(\rho_t, t \in [0, T])$  is in  $V^1(\omega, Q_T)$  and admits a trace  $\gamma(\rho)$  in the sense of Definition 1.1 which satisfies the no-permeability boundary condition (1.4).*

The precise definition of the space  $V^1(\omega, Q_T)$  is given in the Notation section below. The paper is organized as follows. In Section 2 we set the *linear* basis of our approach: we construct the solution to the confined linear Langevin process, obtained by choosing  $b \equiv 0$  in (1.1) and we study the property of its semi-group. This latter will rely on a Feynman–Kac interpretation of the semi-group and the analysis of the boundary value problem:

$$\begin{cases} \partial_t f(t, x, u) - (u \cdot \nabla_x f(t, x, u)) - \frac{\sigma^2}{2} \Delta_u f(t, x, u) = 0, \quad \forall (t, x, u) \in (0, T] \times \mathcal{D} \times \mathbb{R}^d, \\ \lim_{t \rightarrow 0^+} f(t, x, u) = f_0(x, u), \quad \forall (x, u) \in \mathcal{D} \times \mathbb{R}^d, \\ f(t, x, u) = q(t, x, u), \quad \forall (t, x, u) \in \Sigma_T^+, \end{cases} \quad (1.7)$$

for which we prove the existence of a smooth solution, continuous at the boundary (see Theorem 2.6). To the best of our knowledge this results has not yet considered in the PDE literature.

In Section 3, using a PDE approach, we construct a set of time-marginal densities related to a solution to (1.1) as a weak solution to the following nonlinear Vlasov–Fokker–Planck equation with the specular boundary condition (1.3):

$$\begin{cases} \partial_t \rho + (u \cdot \nabla_x \rho) + (B[\cdot; \rho] \cdot \nabla_u \rho) - \frac{\sigma^2}{2} \Delta_u \rho = 0, \quad \text{on } (0, T) \times \mathcal{D} \times \mathbb{R}^d, \\ \rho(0, x, u) = \rho_0(x, u), \quad \text{on } \mathcal{D} \times \mathbb{R}^d, \\ \gamma(\rho)(t, x, u) = \gamma(\rho)(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), \quad \text{on } (0, T] \times \partial \mathcal{D} \times \mathbb{R}^d, \end{cases}$$

where  $\gamma(\rho)$  stands for the trace of  $\rho$  in the sense of Definition 1.1 (see Theorem 3.3 for the existence result). In particular, the verification of the properties (1.5a) and (1.5b) are obtained thank to the construction of Maxwellian bounds for the solution to the nonlinear Vlasov–Fokker–Planck and its trace at the boundary. Starting from this solution, we set a drift  $B(t, x) = B[x; \rho]$  from (1.2) and we construct a process candidate to be a solution of (1.1) using a change of probability measure from the confined Langevin law constructed in Section 2. We achieve the proof of Theorem 1.2 in Section 4, by proving that the resulting set of time-marginal densities coincides with the solution to the Vlasov–Fokker–Planck equation considered in Section 3. We also prove the uniqueness in law for the solution of (1.1).

## 1.2 Notation

For all  $t \in (0, T]$ , we introduce the time-phase space

$$Q_t := (0, t) \times \mathcal{D} \times \mathbb{R}^d,$$

and the boundary sets:

$$\begin{aligned} \Sigma^+ &:= \{(x, u) \in \times \partial \mathcal{D} \times \mathbb{R}^d \text{ s.t. } (u \cdot n_{\mathcal{D}}(x)) > 0\}, & \Sigma_t^+ &:= (0, t) \times \Sigma^+, \\ \Sigma^- &:= \{(x, u) \in \times \partial \mathcal{D} \times \mathbb{R}^d \text{ s.t. } (u \cdot n_{\mathcal{D}}(x)) < 0\}, & \Sigma_t^- &:= (0, t) \times \Sigma^-, \\ \Sigma^0 &:= \{(x, u) \in \partial \mathcal{D} \times \mathbb{R}^d \text{ s.t. } (u \cdot n_{\mathcal{D}}(x)) = 0\}, & \Sigma_t^0 &:= (0, t) \times \Sigma^0, \end{aligned}$$

and further  $\Sigma_T := \Sigma_T^+ \cup \Sigma_T^0 \cup \Sigma_T^- = (0, T) \times \partial\mathcal{D} \times \mathbb{R}^d$ . Denoting by  $d\sigma_{\partial\mathcal{D}}$  the surface measure on  $\partial\mathcal{D}$ , we introduce the product measure

$$d\lambda_{\Sigma_T} := dt \otimes d\sigma_{\partial\mathcal{D}}(x) \otimes du$$

on  $\Sigma_T$ . We set the Sobolev space

$$\mathcal{H}(Q_t) = L^2((0, t) \times \mathcal{D}; H^1(\mathbb{R}^d))$$

equipped with the norm  $\|\cdot\|_{\mathcal{H}(Q_t)}$  defined by

$$\|\phi\|_{\mathcal{H}(Q_t)}^2 = \|\phi\|_{L^2(Q_t)}^2 + \|\nabla_u \phi\|_{L^2(Q_t)}^2.$$

We denote by  $\mathcal{H}'(Q_t)$ , the dual space of  $\mathcal{H}(Q_t)$ , and by  $(\cdot, \cdot)_{\mathcal{H}'(Q_t), \mathcal{H}(Q_t)}$ , the inner product between  $\mathcal{H}'(Q_t)$  and  $\mathcal{H}(Q_t)$ .

We define the weighted Lebesgue space

$$L^2(\omega, Q_t) := \{\psi : Q_t \rightarrow \mathbb{R} ; \sqrt{\omega}\psi \in L^2(Q_t)\},$$

with the weight function  $u \mapsto \omega(u)$  on the velocity variable

$$\omega(u) := (1 + |u|^2)^{\frac{\alpha}{2}}, \text{ for } \alpha > d \vee 2. \quad (1.8)$$

We endow  $L^2(\omega, Q_t)$  with the norm  $\|\cdot\|_{L^2(\omega, Q_t)}$  defined by  $\|\phi\|_{L^2(\omega, Q_t)}^2 = \|\sqrt{\omega}\phi\|_{L^2(Q_t)}^2$ .

We introduce the weighted Sobolev space

$$\mathcal{H}(\omega, Q_t) := \{\psi \in L^2(\omega, Q_t) ; |\nabla_u \psi| \in L^2(\omega, Q_t)\}.$$

with the norm  $\|\cdot\|_{\mathcal{H}(\omega, Q_t)}$  defined by

$$\|\phi\|_{\mathcal{H}(\omega, Q_t)}^2 = \|\phi\|_{L^2(\omega, Q_t)}^2 + \|\nabla_u \phi\|_{L^2(\omega, Q_t)}^2.$$

Finally, we define the set

$$V_1(\omega, Q_T) = \mathcal{C}([0, T]; L^2(\omega, \mathcal{D} \times \mathbb{R}^d)) \cap \mathcal{H}(\omega, Q_T),$$

equipped with the norm

$$\|\phi\|_{V_1(\omega, Q_T)}^2 = \max_{t \in [0, T]} \left\{ \int_{\mathcal{D} \times \mathbb{R}^d} \omega(u) |\phi(t, x, u)|^2 dx du \right\} + \int_{Q_T} \omega(u) |\nabla_u \phi(t, x, u)|^2 dt dx du.$$

We further introduce the spaces

$$\begin{aligned} L^2(\Sigma_T^\pm) &= \{\psi : \Sigma_T^\pm \rightarrow \mathbb{R}; \int_{\Sigma_T^\pm} |(u \cdot n_{\mathcal{D}}(x))| |\psi(t, x, u)|^2 d\lambda_{\Sigma_T}(t, x, u) < +\infty\}, \\ L^2(\omega, \Sigma_T^\pm) &= \{\psi : \Sigma_T^\pm \rightarrow \mathbb{R}; \int_{\Sigma_T^\pm} \omega(u) |(u \cdot n_{\mathcal{D}}(x))| |\psi(t, x, u)|^2 d\lambda_{\Sigma_T}(t, x, u) < +\infty\}, \end{aligned}$$

equipped with their respective norms

$$\begin{aligned} \|\psi\|_{L^2(\Sigma_T^\pm)} &= \sqrt{\int_{\Sigma_T^\pm} |(u \cdot n_{\mathcal{D}}(x))| |\psi(t, x, u)|^2 d\lambda_{\Sigma_T}(t, x, u)}, \\ \|\psi\|_{L^2(\omega, \Sigma_T^\pm)} &= \sqrt{\int_{\Sigma_T^\pm} \omega(u) |(u \cdot n_{\mathcal{D}}(x))| |\psi(t, x, u)|^2 d\lambda_{\Sigma_T}(t, x, u)}. \end{aligned}$$

## 2 Preliminaries: the confined Langevin process

In this section, we prove the well-posedness of the confined linear Langevin equation :

$$\begin{cases} X_t = x_0 + \int_0^t U_s ds, \\ U_t = u_0 + \sigma W_t + K_t, \\ K_t = -2 \sum_{0 \leq s \leq t} (U_{s-} \cdot n_{\mathcal{D}}(X_s)) n_{\mathcal{D}}(X_s) \mathbb{1}_{\{X_s \in \partial \mathcal{D}\}}, \quad \forall t \in [0, T], \end{cases} \quad (2.1)$$

for any  $(x_0, u_0) \in (\mathcal{D} \times \mathbb{R}^d) \cup \Sigma \setminus \Sigma^0$ . We further investigate some properties (notably the  $L^p$ -stability) of its semigroup. The verification of the mean no-permeability boundary condition (1.4) will be treated in Section 4.

### 2.1 Existence and uniqueness

We focus our well-posedness result to the case where  $(x_0, u_0) \in (\mathcal{D} \times \mathbb{R}^d) \cup \Sigma^-$ , which is the situation where either the particle starts from the interior  $\mathcal{D}$  or starts from the boundary with an ingoing velocity. As a natural extension to the case where  $(x_0, u_0) \in \Sigma^+$  (namely the situation of an initial outgoing velocity) with the flow notation, we define the solution of (2.1) starting from  $(x_0, u_0) \in \Sigma^+$  by

$$((X_t, U_t))^{0, x_0, u_0}, t \in [0, T] = \left( (X_t, U_t))^{0, x_0, u_0 - 2(u_0 \cdot n_{\mathcal{D}}(x_0)) n_{\mathcal{D}}(x_0)}, t \in [0, T] \right).$$

The construction presented hereafter takes advantage of the regularity of  $\partial \mathcal{D}$  to locally straighten the boundary, in the same manner as the diffracted process across a submanifold have been constructed in [4]. This allows us to adapt the one dimensional construction proposed in [5] which was based on the explicit law of the sequence of passage times at zero of the 1D-Brownian motion primitive presented in [24], [20]. Our main result is the following:

**Theorem 2.1.** *Under  $(H_{\text{Langevin}})$ -(ii), for any  $(x_0, u_0) \in (\mathcal{D} \times \mathbb{R}^d) \cup \Sigma^-$ , there exists a weak solution to (2.1). Moreover, the sequence of hitting times*

$$\tau_n = \inf\{t > \tau_{n-1} ; X_t \in \partial \mathcal{D}\}, \text{ for } n \geq 1, \tau_0 = 0,$$

*is well defined and grows to infinity. The pathwise uniqueness holds for the solution of (2.1).*

For the sake of completeness, we recall some results related to the local straightening of the boundary  $\partial \mathcal{D}$  as given in [4]. Since  $\partial \mathcal{D}$  is smooth, one can construct a mapping  $\pi$  of class  $C_b^2$  from a neighborhood  $\mathcal{N}$  of  $\partial \mathcal{D}$  to  $\partial \mathcal{D}$  such that

$$|x - \pi(x)| = d(x, \partial \mathcal{D}), \quad \forall x \in \mathcal{N},$$

where  $d(x, \partial \mathcal{D})$  denotes the distance between  $x$  and the set  $\partial \mathcal{D}$ . Note that, reducing  $\mathcal{N}$  if necessary, we can always assume that  $\pi$  is  $C_b^2(\overline{\mathcal{N}})$ . For all  $x \in \mathcal{N}$ , set

$$\sigma(x) := (x - \pi(x)) \cdot n_{\mathcal{D}}(\pi(x)), \quad (2.2)$$

so that  $\sigma(x)$  is the signed distance to  $\partial \mathcal{D}$  (positive in  $\mathbb{R}^d \setminus \overline{\mathcal{D}}$ , negative in  $\mathcal{D}$ ) and is of class  $C_b^2(\mathcal{N})$ . We still denote by  $\sigma$  a  $C_b^2(\mathbb{R}^d)$  extension of this function to the whole Euclidean space. It is well-known that

$$\nabla \sigma(x) = n_{\mathcal{D}}(\pi(x)), \quad \forall x \in \mathcal{N} \quad (2.3)$$

(see e.g. [18, p. 355]).

**Proposition 2.2** ([4], Proposition 2.1). *Under  $(H_{\text{Langevin}})$ -(ii), there exists a family of bounded open subsets of  $\mathcal{N}$ ,  $\{\mathcal{U}_1, \dots, \mathcal{U}_{M-1}\}$  such that  $\partial \mathcal{D} \subset \cup_{i=1}^{M-1} \mathcal{U}_i$ , and a family of  $\mathbb{R}^d$ -valued functions  $\{\psi_1, \dots, \psi_{M-1}\}$  such that, for all  $1 \leq i \leq M-1$ ,  $\psi_i = (\psi_i^{(1)}, \dots, \psi_i^{(d)})$  is a  $C_b^2$  diffeomorphism from  $\mathcal{U}_i$  to  $\psi_i(\mathcal{U}_i)$ , admitting a  $C_b^2$  extension on  $\overline{\mathcal{U}_i}$  and satisfying for all  $x \in \overline{\mathcal{U}_i}$*

$$\begin{cases} \psi_i^{(d)}(x) = \sigma(x), \\ \nabla \psi_i^{(k)}(x) \cdot n_{\mathcal{D}}(\pi(x)) = 0, \quad \forall k \in \{1, 2, \dots, d-1\}, \\ \frac{\partial \psi_i^{-1}}{\partial x_d}(\psi_i(x)) = n_{\mathcal{D}}(\pi(x)). \end{cases} \quad (2.4)$$

Note that, by (2.4),  $\psi_i(\mathcal{U}_i \cap \partial\mathcal{D}) \subset \mathbb{R}^{d-1} \times \{0\}$ , which justifies the term “local straightening”.

Let  $\mathcal{U}_M$  be an open subset of  $\mathbb{R}^d$  such that  $\partial\mathcal{D} \cap \mathcal{U}_M = \emptyset$  and  $\cup_{i=1}^M \mathcal{U}_i = \mathbb{R}^d$ , and set  $\bar{\psi}_M(x) := x$  on  $\mathcal{U}_M$ .

*Proof of Theorem 2.1.* For any  $(x_0, u_0) \in (\mathcal{D} \times \mathbb{R}^d) \cup \Sigma^-$ , we consider the flow of processes  $((x_t, u_t)^{s, x_0, u_0}, s \leq t \leq T)$  in  $\mathbb{R}^{2d}$ , defined by

$$\begin{cases} x_t^{s, x_0, u_0} = x_0 + \int_s^t u_s^{s, x_0, u_0} ds, \\ u_t^{s, x_0, u_0} = u_0 + \sigma W_t - W_s. \end{cases} \quad (2.5)$$

For notation convenience, we set  $(x_t, u_t) := (x_t, u_t)^{0, x_0, u_0}$ . We introduce the index  $i_1$  that corresponds to the smallest index of the open subsets for which  $x_0$  is the most “deeply” contained:

$$i_1 = \inf \left\{ 1 \leq j \leq M; d(x_0, \mathbb{R}^d \setminus \mathcal{U}_j) = \sup_{1 \leq m \leq M} d(x_0, \mathbb{R}^d \setminus \mathcal{U}_m) \right\}.$$

We consider also the exit time

$$\zeta_1 = \inf \{t \geq 0; x_t \notin \mathcal{U}_{i_1}\}.$$

If  $i_1 = M$ , we set for all  $t \leq \zeta_1$ ,

$$(X_t, U_t) = (x_t, u_t).$$

Else, suppose that for each  $i = 1, \dots, M-1$ , the diffeomorphism  $\psi_i$  on  $\mathcal{U}_i$  admits a  $C^2$  extension on  $\mathbb{R}^d$ , satisfying (2.4). Applying the Itô formula to the vector  $(Y_t, V_t) = ((Y_t^{(k)}, V_t^{(k)}); k = 1, \dots, d)$  given by

$$(Y_t^{(k)}, V_t^{(k)}) := \left( \psi_{i_1}^{(k)}(x_t), (\nabla_x \psi_{i_1}^{(k)}(x_t) \cdot u_t) \right) = \left( \psi_{i_1}^{(k)}(x_t), \sum_{l=1}^d \partial_{x_l} \psi_{i_1}^{(k)}(x_t) u_t^{(l)} \right),$$

for all  $0 \leq t \leq \zeta_1$ , we obtain that  $(Y, V)$  is a solution to the following SDE

$$\begin{cases} Y_t^{(k)} = \psi_{i_1}^{(k)}(x_0) + \int_0^t V_s^{(k)} ds, \\ V_t^{(k)} = \left( \nabla_x \psi_{i_1}^{(k)}(x_0) \cdot u_0 \right) + \sigma \int_0^t \sum_{l=1}^d \partial_{x_l} \psi_{i_1}^{(k)}(\psi_{i_1}^{-1}(Y_s)) dW_s^{(l)}, \\ \quad + \int_0^t \mathbb{1}_{\{s \leq \zeta_1\}} \sum_{1 \leq l, n \leq d} \partial_{x_l, x_n}^2 \psi_{i_1}^{(k)}(\psi_{i_1}^{-1}(Y_s)) (\nabla_x \psi_{i_1}^{-1}(Y_s) V_s)^{(l)} (\nabla_x \psi_{i_1}^{-1}(Y_s) V_s)^{(n)} ds. \end{cases} \quad (2.6)$$

The SDE above has a non-homogeneous diffusion coefficient and a drift coefficient with quadratic growth. Nevertheless, since  $\max_{t \in [0, T]} |u_t|^2$  is finite  $\mathbb{P}$ -a.s., the same holds true for  $\max_{t \in [0, T]} |V_t \wedge \zeta_1|^2$  so that the solution does not explode at finite time and is pathwise unique.

Note that, from (2.4) and (2.3),  $\|\nabla_x \psi_{i_1}^{(d)}(x)\| = 1$  so that the diffusion term in  $V_t^{(d)}$  is a local martingale with a quadratic variation given by

$$\left\langle \sum_{l=1}^d \partial_{x_l} \int_0^\cdot \partial_{x_l} \psi_{i_1}^{(d)}(\psi_{i_1}^{-1}(Y_s)) dW_s^{(l)} \right\rangle_t = t.$$

By Levy characterization, it follows that

$$\widetilde{W}_t = \int_0^t \sum_{l=1}^d \partial_{x_l} \psi_{i_1}^{(d)}(\psi_{i_1}^{-1}(Y_s)) dW_s^{(l)}$$

is a standard Brownian motion in  $\mathbb{R}^d$ . Now, from the identity  $\psi^{-1}(\psi(x)) = x$ , we easily derive that  $\sum_{i=1}^d \partial_{x_i} (\psi^{-1})^{(k)}(\psi(x)) \partial_{x_l} \psi^{(i)}(x) = \delta_{kl}$ , where  $\delta_{kl}$  is the Kronecker delta, and

$$(\psi_{i_1}^{-1}(Y_t), \nabla_x \psi_{i_1}^{-1}(Y_t) V_t) = (x_t, u_t).$$

Then for any component  $k$ , the drift term

$$\int_0^t \mathbb{1}_{\{s \leq \zeta_1\}} \sum_{1 \leq l, n \leq d} \partial_{x_l, x_n}^2 \psi_{i_1}^{(k)}(\psi_{i_1}^{-1}(Y_s)) (\nabla_x \psi_{i_1}^{-1}(Y_s) V_s)^{(l)} (\nabla_x \psi_{i_1}^{-1}(Y_s) V_s)^{(n)} ds$$

of  $V_t^{(k)}$  is such that

$$\begin{aligned} & \mathbb{E} \left[ \int_0^t \left( \mathbb{1}_{\{s \leq \zeta_1\}} \sum_{1 \leq l, n \leq d} \partial_{x_l, x_n}^2 \psi_{i_1}^{(k)}(\psi_{i_1}^{-1}(Y_s)) (\nabla_x \psi_{i_1}^{-1}(Y_s) V_s)^{(l)} (\nabla_x \psi_{i_1}^{-1}(Y_s) V_s)^{(n)} \right)^2 ds \right] \\ &= \mathbb{E} \left[ \int_0^t \left( \mathbb{1}_{\{s \leq \zeta_1\}} \sum_{1 \leq l, n \leq d} \partial_{x_l, x_n}^2 \psi_{i_1}^{(k)}(x_s) u_s^{(l)} u_s^{(n)} \right)^2 ds \right] \\ &\leq \sup_{1 \leq l, n \leq d} \|\partial_{x_l, x_n}^2 \psi_{i_1}^{(k)}\|_{L^\infty(\mathcal{U}_{i_1})} \mathbb{E} \left[ \int_0^t \left( u_s^{(l)} u_s^{(n)} \right)^2 ds \right] < +\infty. \end{aligned}$$

Consequently (see [Lipter–Shiryaev \[22, Theorem 7.4\]](#)), the law of  $(Y_t, V_t, t \in [0, T])$  is absolutely continuous w.r.t. the law of  $(\mathcal{Y}_t, \mathcal{V}_t, t \in [0, T])$ , solution to

$$\begin{cases} \mathcal{Y}_t^{(k)} = \psi_{i_1}^{(k)}(x_0) + \int_0^t \mathcal{V}_s^{(k)} ds, \\ \mathcal{V}_t^{(k)} = \left( \sum_{l=1}^d \partial_{x_l} \psi_{i_1}^{(k)}(x_0) u_0^{(l)} \right) + \sigma \widetilde{W}_s. \end{cases} \quad (2.7)$$

In particular, McKean [\[24\]](#) has shown that if  $(Y_0^d, V_0^d) \neq (0, 0)$  then,  $\mathbb{P}$ -almost surely, the paths  $t \mapsto (\mathcal{Y}_t^{(d)}, \mathcal{V}_t^{(d)})$  never cross  $(0, 0)$ . Thus the sequence of passage times at zero of  $(\mathcal{Y}_t^{(d)})$  tends to infinity, and the same holds true for the sequence of passage times at zero  $(\beta_n^1, n \geq 0)$  of  $(Y_t^{(d)})$  as well as for the sequence of hitting times of  $\partial\mathcal{D}$  by  $(x_t)$ . We set

$$(X_t, U_t) = (x_t, u_t) \text{ for all } 0 \leq t < \beta_1^1 \wedge \zeta_1.$$

Suppose that  $\beta_1^1 < \zeta_1$ . At time  $\beta_1^1$ , as  $x_{\beta_1^1} \in \partial\mathcal{D}$  with  $(u_{\beta_1^1} \cdot n_{\mathcal{D}}(x_{\beta_1^1})) > 0$   $\mathbb{P}$ -a.s., we reflect the velocity as follows:

$$(X_{\beta_1^1}, U_{\beta_1^1}) = \left( x_{\beta_1^1}, u_{\beta_1^1-} - 2(u_{\beta_1^1-} \cdot n_{\mathcal{D}}(x_{\beta_1^1})) n_{\mathcal{D}}(x_{\beta_1^1}) \right).$$

We resume the first step of our construction: we set  $\theta_0 = 0$  and we have defined

$$\begin{aligned} i_1 &= \inf \left\{ 1 \leq j \leq M; d(X_{\theta_0}, \mathbb{R}^d \setminus \mathcal{U}_j) = \sup_{1 \leq m \leq M} d(X_{\theta_0}, \mathbb{R}^d \setminus \mathcal{U}_m) \right\}, \\ \zeta_1 &= \inf \{ t \geq \theta_0; x_t \notin \mathcal{U}_{i_1} \}, \\ \beta_1^1 &= \inf \{ t \geq \theta_0; x_t \in \partial\mathcal{D} \}. \end{aligned}$$

We set  $\theta_1 = \beta_1^1 \wedge \zeta_1$  and

$$\begin{aligned} & (X_t, U_t) = (x_t, u_t) \text{ for all } \theta_0 \leq t < \theta_1, \\ & \text{and } (X_{\theta_1}, U_{\theta_1}) = \left( x_{\theta_1}, u_{\theta_1-} - 2(u_{\theta_1-} \cdot n_{\mathcal{D}}(x_{\theta_1})) n_{\mathcal{D}}(x_{\theta_1}) \mathbb{1}_{\{x_{\theta_1} \in \partial\mathcal{D}\}} \right). \end{aligned}$$

We iterate the procedure as follows: assume that we have construct the process  $(X_t, U_t)$  on  $[0, \theta_n \wedge T]$ . We define

$$i_{n+1} = \inf \left\{ 1 \leq j \leq M; d(X_{\theta_n}, \mathbb{R}^d \setminus \mathcal{U}_j) = \sup_{1 \leq m \leq M} d(X_{\theta_n}, \mathbb{R}^d \setminus \mathcal{U}_m) \right\}, \quad (2.8)$$

$$\zeta_{n+1} = \inf \{ t \geq \theta_n; x_t \notin \mathcal{U}_{i_{n+1}} \}, \quad (2.9)$$

$$\beta_1^{n+1} = \inf \{ t \geq \theta_n; x_t \in \partial\mathcal{D} \}. \quad (2.10)$$

where  $(x, u)$  denotes now the solution of Equation (2.5) starting at  $(\theta_n, X_{\theta_n}, U_{\theta_n})$ . We set

$$\theta_{n+1} = \beta_1^{n+1} \wedge \zeta_{n+1}, \quad (2.11)$$

$$(X_t, U_t) = (x_t, u_t)^{\theta_n, X_{\theta_n}, U_{\theta_n}} \text{ for all } \theta_n \leq t < \theta_{n+1}, \quad (2.12)$$

$$\text{and } (X_{\theta_{n+1}}, U_{\theta_{n+1}}) = \left( x_{\theta_{n+1}}, u_{\theta_{n+1}}^- - 2(u_{\theta_{n+1}}^- \cdot n_{\mathcal{D}}(x_{\theta_{n+1}}))n_{\mathcal{D}}(x_{\theta_{n+1}})\mathbb{1}_{\{X_{\theta_{n+1}} \in \partial\mathcal{D}\}} \right). \quad (2.13)$$

By construction  $(X_t, U_t)$  is solution of (2.1) and the sequence of hitting times  $\{\tau_n, n \geq 0\}$  is well defined as each  $\tau_n$  corresponds to some  $\beta_1^m$ . We conclude on the existence of a solution to (2.1) with the following lemma proved below.

**Lemma 2.3.**  $\mathbb{P}_{(x_0, u_0)}$ -a.s., the sequence  $\{\theta_n, n \in \mathbb{N}\}$  given in (2.11) grows to infinity as  $n$  tends to  $\infty$ .

We also emphasize that, starting from  $(x_0, u_0) \in (\mathcal{D} \times \mathbb{R}^d) \cup \Sigma^-$ ,  $\mathbb{P}_{(x_0, u_0)}$ -almost surely any path of solution of (2.5) never cross  $\Sigma^0$ . Indeed, the probability for the solution of (2.5) to cross  $\Sigma^0$  is dominated by the probability that a piece of straightened path (starting in  $\mathcal{D}$ ) crosses  $(\mathbb{R}^{d-1} \times \{0\})^2$  which is nil by the McKean result.

The pathwise uniqueness result is then a consequence of the above remark. Consider  $(X, U)$  ( $\tilde{X}, \tilde{U}$ ), two solutions to (2.1) defined on the same probability space, endowed with the same Brownian motion. Since  $x_0 \in \mathcal{D}$ , we can define the first passage time at zero  $\tilde{\tau}_1$  of  $\tilde{X}$ , and we observe that  $\tilde{\tau}_1 = \tau_1$  due to the continuity of  $X$  and  $\tilde{X}$ . It follows that  $U_{\tau_1 \wedge \tilde{\tau}_1} = \tilde{U}_{\tau_1 \wedge \tilde{\tau}_1}$ , so that  $(X, U)$  and  $(\tilde{X}, \tilde{U})$  are equal up to  $\tau_1$ . By induction, one checks that this assertion holds true up to  $\tau_n$  for all  $n \in \mathbb{N}$ . As  $\tau_n$  tends to  $+\infty$   $\mathbb{P}_{(x_0, u_0)}$ -a.s.,  $(X, U)$  and  $(\tilde{X}, \tilde{U})$  are equal on  $[0, T]$ .  $\square$

*Proof of Lemma 2.3.* We already know that the sequence  $\{\beta_1^n; n \in \mathbb{N}\}$  is well defined and grows to infinity. So we only have to prove that  $\{\zeta_n; n \in \mathbb{N}\}$  grows to infinity. We simplify the presentation of the proof by considering the sequence of stopping times  $\{\beta_1^n; n \in \mathbb{N}\}$  without interlacing  $\{\beta_1^n; n \in \mathbb{N}\}$ :

$$i_{n+1} = \inf \left\{ 1 \leq j \leq M; d(x_{\zeta_n}, \mathbb{R}^d \setminus \mathcal{U}_j) = \sup_{1 \leq m \leq M} d(x_{\zeta_n}, \mathbb{R}^d \setminus \mathcal{U}_m) \right\}, \quad (2.14)$$

$$\zeta_{n+1} = \inf \{ t \geq \zeta_n; x_t \notin \mathcal{U}_{i_{n+1}} \}, \quad (2.15)$$

where  $(x_t, u_t, t \in [0, +\infty))$  is the solution of (2.5).

For any  $i \in \{1, \dots, M\}$  and any  $x \in \partial\mathcal{U}_i$ ,  $\sup_{j \neq i} d(x, \mathbb{R}^d \setminus \mathcal{U}_j)$  is positive and continuous with respect to  $x \in \partial\mathcal{U}_i$ . Since  $\mathcal{U}_i$  is bounded for  $1 \leq i \leq M-1$  and  $\partial\mathcal{U}_M \subset \bigcup_{i=1}^{M-1} \mathcal{U}_i$ , the set  $\partial\mathcal{U}_i$  is compact for any  $i \in \{1, \dots, M\}$ . Hence, we can define the strictly positive constant  $\gamma_0$  as the minimal distance that allows the process  $(x_t)$  to go from one  $\mathcal{U}_i$  to another  $\mathcal{U}_j$ ,

$$\gamma_0 := \inf_{1 \leq i \leq M} \inf_{x \in \partial\mathcal{U}_i} \sup_{j \neq i} d(x, \mathbb{R}^d \setminus \mathcal{U}_j) > 0.$$

The idea is to prove that, almost surely,  $\zeta_k - \zeta_{k-1} \geq \mathcal{T}$  infinitely often w.r.t.  $k \geq 1$  for  $\mathcal{T}$  small enough. We fix a  $\mathcal{T} > 0$  and we consider  $A_k = \{\sup_{0 \leq t \leq \mathcal{T}} |x_{\zeta_{k-1}+t} - x_{\zeta_{k-1}}| \leq \gamma_0\}$ , such that

$$\mathbb{P}(A_n^c | \mathcal{F}_{\zeta_{n-1}}) \leq \frac{\mathcal{T}}{\gamma_0} \mathbb{E} \sup_{0 \leq t \leq \mathcal{T}} |u_t|.$$

Since  $A_k \in \mathcal{F}_{\zeta_k}$  for all  $k \geq 1$ , this implies that, for all  $m < n$ ,

$$\begin{aligned} \mathbb{P}\left(\bigcap_{m \leq k \leq n} A_k^c\right) &= \mathbb{E}\left(\mathbb{1}_{\bigcap_{m \leq k \leq n-1} A_k^c} \mathbb{P}(A_n^c | \mathcal{F}_{\zeta_{n-1}})\right) \\ &\leq \mathbb{P}\left(\bigcap_{m \leq k \leq n-1} A_k^c\right) \frac{\mathcal{T}}{\gamma_0} \mathbb{E} \sup_{0 \leq t \leq \mathcal{T}} |u_t| \leq \dots \leq \left(\frac{\mathcal{T}}{\gamma_0} \mathbb{E} \sup_{0 \leq t \leq \mathcal{T}} |u_t|\right)^{n-m+1}. \end{aligned}$$

Therefore, choosing  $\mathcal{T}$  such that  $\frac{\mathcal{T}}{\gamma_0} \mathbb{E} \sup_{0 \leq t \leq \mathcal{T}} |u_0 + \sigma W_t| < 1$ , it follows that

$$\lim_{n \rightarrow +\infty} \mathbb{P}\left(\bigcap_{m \leq k \leq n} A_k^c\right) = 0, \quad \forall m \geq 1.$$

This entails that the events  $A_k$  occur infinitely often  $\mathbb{P}$ -a.s. We thus found  $\mathcal{T} > 0$  such that the events  $\{\zeta_k - \zeta_{k-1} \geq \mathcal{T}\}$  a.s. occur infinitely often.  $\square$

The following lemma is used in Section 4 to achieve the construction of the nonlinear process.

**Lemma 2.4.** *Let  $(X, U)$  be the confined Langevin process solution to 2.1 and  $\{\tau_n, n \in \mathbb{N}\}$  be the related sequence of hitting times defined in Theorem 2.1. Then, for all  $(x_0, u_0) \in (\mathcal{D} \times \mathbb{R}^d) \cup (\Sigma \setminus \Sigma^0)$ , for all integer  $K > 1$ , the measure  $\sum_{n=0}^K \mathbb{P}_{(x_0, u_0)}(\tau_n \in dt, X_{\tau_n} \in dx, U_{\tau_n} \in du)$  is absolutely continuous w.r.t the measure  $\lambda_\Sigma(dt, dx, du)$ .*

*Proof.* Owing to the strong Markov property of  $(X, U)$ , it is sufficient to prove that for all  $(x_0, u_0) \in \Sigma^-$ , the probability  $\mathbb{P}_{(x_0, u_0)} \circ (\tau_1, X_{\tau_1}, U_{\tau_1})^{-1}$  is absolutely continuous w.r.t.  $\lambda_\Sigma$ . Using the same notation than in the proof of Theorem 2.1, for  $(x_0, u_0)$  corresponding to some  $(x_{\zeta_n}, u_{\zeta_n})$  in the iterative construction ((2.8)–(2.13)), it is enough to prove that  $\mathbb{P}_{(x_0, u_0)} \circ (\beta_1^1, x_{\beta_1^1}, u_{\beta_1^1})^{-1}$  and  $\lambda_\Sigma$  are equivalent, where  $(x_t, u_t, t \in [0, T])$  is the solution of (2.5). Let  $i_1$  be the index of the subset  $\mathcal{U}_{i_1}$  such that  $x_{\beta_1^1}^{(x_0, u_0)} \in \mathcal{U}_{i_1} \cap \partial\mathcal{D}$ ,  $\psi_{i_1}(x_t) = Y_t$ ,  $(\nabla_x \psi_{i_1}(x_t) \cdot u_t) = V_t$  and  $\beta_1^1 = \inf\{t > 0; Y_t^{(d)} = 0\}$  (with  $(Y_t, V_t, t \in [0, T])$  solution to (2.6)). Then any measurable test function  $f$ ,

$$\mathbb{E}_{\mathbb{P}_{(x_0, u_0)}} \left[ f(\beta_1^1, x_{\beta_1^1}, u_{\beta_1^1}) \right] = \mathbb{E}_{\mathbb{P}_{(y_0, v_0)}} \left[ f(\beta_1^1, \psi_{i_1}^{-1}(Y_{\beta_1^1}), (\nabla_y \psi_{i_1}^{-1}(Y_{\beta_1^1}) \cdot V_{\beta_1^1})) \right],$$

where  $(y_0, v_0) = (\psi_{i_1}^{-1}(x_0), (\nabla_y \psi_{i_1}^{-1} \cdot u_0))$ . At this point, and owing to the equivalence between the laws of  $(Y_t, V_t, t \in [0, T])$  and  $(\mathcal{Y}_t, \mathcal{V}_t, t \in [0, T])$ , solution to (2.7), we are reduce to prove that the law of  $(\beta_1^1, \psi_{i_1}^{-1}(\mathcal{Y}_{\beta_1^1}), (\nabla_y \psi_{i_1}^{-1}(\mathcal{Y}_{\beta_1^1}) \cdot V_{\beta_1^1}))$  is absolutely continuous w.r.t.  $\lambda_\Sigma$ . Let us first recall that the joint law of  $(\beta_1^1, \mathcal{Y}_{\beta_1^1}^{(d)}, \mathcal{V}_{\beta_1^1}^{(d)})$  is explicitly known (see [20, Theorem 1]) and is absolutely continuous w.r.t. the product measure  $dt \otimes du^{(d)}$ . Furthermore,  $\beta_1^1$  is independent of the  $(d-1)$ -first components  $(\mathcal{Y}_t', \mathcal{V}_t', t \in [0, T])$ . Hence we remark that the law of  $(\beta_1^1, \mathcal{Y}_{\beta_1^1}', \mathcal{V}_{\beta_1^1}')$  is absolutely continuous w.r.t.  $dt \otimes dy' \otimes du$ .

Let us next recall the characterization of the surface measure  $\sigma_{\partial\mathcal{D}}$  (see e.g [1, Chapter 5]). As  $\partial\mathcal{D}$  is  $\mathcal{C}^3$ , it can be (locally) represented as the graph of a  $\mathcal{C}^3$  function: for all  $x \in \partial\mathcal{D}$ , there exists an open neighborhood  $\mathcal{U}_x \subset \mathbb{R}^d$  of  $x$  and a  $\mathcal{C}_b^3(\mathbb{R}^{d-1})$  function  $\phi_x$ , such that for all  $y \in \partial\mathcal{D} \cap \overline{\mathcal{U}_x}$ ,  $y^{(d)} = \phi_x(y^{(1)}, \dots, y^{(d-1)})$  and  $\mathcal{D} \cap \mathcal{U}_x = \{y = (y', y^{(d)}) \in \mathcal{U}_x, \text{ s.t } y^{(d)} < \phi_x(y')\}$ . Hence, for all  $\mathcal{C}^\infty$ -function  $g$  with compact support in  $\partial\mathcal{D} \cap \overline{\mathcal{U}_x}$ , we have that

$$\langle \sigma_{\partial\mathcal{D}}, g \rangle = \int_{\mathbb{R}^{d-1}} g((y', \phi_x(y'))) \sqrt{1 + |\nabla \phi_x(y')|^2} dy'.$$

Owing to this characterization and the preceding remark on the law of  $(\beta_1^1, \mathcal{Y}_{\beta_1^1}', \mathcal{V}_{\beta_1^1}')$ , we conclude on the result.  $\square$

## 2.2 On the semigroup of the confined Langevin process

In this section we investigate some estimates related to the semigroup associated to the solution  $(X_t, U_t)$  of (2.1); namely, for some nonnegative test function  $\psi : \mathcal{D} \times \mathbb{R}^d \rightarrow \mathbb{R}$ , for all  $(x, u) \in (\mathcal{D} \times \mathbb{R}^d) \cup (\Sigma \setminus \Sigma^0)$ , we define

$$\Gamma^\psi(t, x, u) := \mathbb{E}_{\mathbb{P}_{(x, u)}} [\psi(X_t, U_t)]. \quad (2.16)$$

As in [5], this semigroup is of particular interest for the construction of the solution to the nonlinear equation (1.1). Due to the pathwise uniqueness of the confined Langevin process, one has that, for all  $0 \leq s \leq t \leq T$ ,

$$\Gamma^\psi(t-s, x, u) = \mathbb{E}_{\mathbb{P}_{s, (x, u)}} [\psi(X_t, U_t)], \quad (2.17)$$

so that the estimates hereafter can be extended to the semigroup transitions of the process.

We consider also the semigroup related to the stopped process, namely

$$\Gamma_n^\psi(t, x, u) = \mathbb{E}_{\mathbb{P}_{(x, u)}} [\psi(X_{t \wedge \tau_n}, U_{t \wedge \tau_n})],$$

$(\tau_n, n \in \mathbb{N})$  being the sequence of hitting times defined in Theorem 2.1 and  $\Gamma_0^\psi(t, x, u) = \psi(x, u)$ .

**Proposition 2.5.** Assume  $(H_{\text{Langevin}})$ . Then, for all nonnegative  $\psi \in \mathcal{C}_c^\infty(\mathcal{D} \times \mathbb{R}^d)$  and all  $n \in \mathbb{N}^*$ ,  $\Gamma_n^\psi$  is  $\mathcal{C}_b^{1,1,2}(Q_T) \cap \mathcal{C}(\overline{Q_T \setminus \Sigma_T^0})$  and satisfies the PDE

$$\begin{cases} \partial_t \Gamma_n^\psi(t, x, u) - (u \cdot \nabla_x \Gamma_n^\psi(t, x, u)) - \frac{\sigma^2}{2} \Delta_u \Gamma_n^\psi(t, x, u) = 0, & \text{for all } (t, x, u) \in Q_T, \\ \Gamma_n^\psi(0, x, u) = \psi(x, u), & \text{for all } (x, u) \in \mathcal{D} \times \mathbb{R}^d, \\ \Gamma_n^\psi(t, x, u) = \Gamma_{n-1}^\psi(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), & \text{for all } (t, x, u) \in \Sigma_T^+. \end{cases} \quad (2.18)$$

In addition,  $\Gamma_n^\psi$  belongs to  $L^2((0, T) \times \mathcal{D}; H^1(\mathbb{R}^d))$  and satisfies the energy equality

$$\|\Gamma_n^\psi(t)\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \sigma^2 \|\nabla_u \Gamma_n^\psi\|_{L^2(Q_t)}^2 + \|\Gamma_n^\psi\|_{L^2(\Sigma_t^-)}^2 = \|\psi\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \|\Gamma_{n-1}^\psi\|_{L^2(\Sigma_t^-)}^2. \quad (2.19)$$

The core of the proof of Proposition 2.5 relies on the following PDE result, the proof of which is postponed in the next section.

**Theorem 2.6.** Assume  $(H_{\text{Langevin}})$ . Given two nonnegative functions  $f_0 \in L^2(\mathcal{D} \times \mathbb{R}^d) \cap \mathcal{C}_b(\mathcal{D} \times \mathbb{R}^d)$  and  $q \in L^2(\Sigma_T^+) \cap \mathcal{C}_b(\Sigma_T^+)$ , there exists a unique nonnegative function  $f \in \mathcal{C}_b^{1,1,2}(Q_T) \cap \mathcal{C}(\overline{Q_T \setminus \Sigma_T^0}) \cap L^2((0, T) \times \mathcal{D}; H^1(\mathbb{R}^d))$  solution to

$$\begin{cases} \partial_t f(t, x, u) - (u \cdot \nabla_x f(t, x, u)) - \frac{\sigma^2}{2} \Delta_u f(t, x, u) = 0, & \text{for all } (t, x, u) \in Q_T, \\ \lim_{t \rightarrow 0^+} f(t, x, u) = f_0(x, u), & \text{for all } (x, u) \in \mathcal{D} \times \mathbb{R}^d, \\ f(t, x, u) = q(t, x, u), & \text{for all } (t, x, u) \in \Sigma_T^+. \end{cases} \quad (2.20)$$

Furthermore, for all  $t \in (0, T)$ ,  $f$  satisfies the energy equality:

$$\|f(t)\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \sigma^2 \|\nabla_u f\|_{L^2(Q_t)}^2 + \|f(t)\|_{L^2(\Sigma_t^-)}^2 = \|f_0\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \|q\|_{L^2(\Sigma_t^+)}^2, \quad (2.21)$$

and the  $L^p$ -estimate:

$$\|f(t)\|_{L^p(\mathcal{D} \times \mathbb{R}^d)}^p + \|f\|_{L^p(\Sigma_t^-)}^p + \sigma^2 p(p-1) \|\nabla_u f\|_{L^p(Q_t)}^p \leq \|f_0\|_{L^p(\mathcal{D} \times \mathbb{R}^d)}^p + \|q\|_{L^p(\Sigma_t^+)}^p. \quad (2.22)$$

*Proof of Proposition 2.5.* Considering  $f_0 = \psi$  and  $q = \psi|_{\Sigma_T^+} = 0$  (since  $\psi$  has its support in the interior of  $\mathcal{D}$ ), Theorem 2.6 ensures that there exists a (classical) solution  $f_1$  to (2.20) on  $Q_T$ . The Feynmann–Kac formula (see e.g. [17]) allows us to identify

$$f_1(t, x, u) = \mathbb{E}_{\mathbb{P}_{(x,u)}} [\psi(X_{t \wedge \tau_1}, U_{t \wedge \tau_1})] = \Gamma_1^\psi(t, x, u).$$

Consequently,  $\Gamma_1^\psi \in \mathcal{C}_b^{1,1,2}(Q_T) \cap \mathcal{C}(\overline{Q_T \setminus \Sigma_T^0}) \cap L^2((0, T) \times \mathcal{D}; H^1(\mathbb{R}^d))$  and is continuous along  $\Sigma_T^-$ .

Consider now,  $\Gamma_{n-1}^\psi \in \mathcal{C}_b^{1,1,2}(Q_T) \cap \mathcal{C}(\overline{Q_T \setminus \Sigma_T^0})$ , for  $n > 1$ . One observes that, for all  $(t, x, u) \in \Sigma_T^+$

$$\begin{aligned} \lim_{\mathcal{D} \times \mathbb{R}^d \ni (y,v) \rightarrow ((x,u))} \Gamma_n^\psi(t, y, v) &= \lim_{\mathcal{D} \times \mathbb{R}^d \ni (y,v) \rightarrow ((x,u))} \mathbb{E}_{\mathbb{P}_{(y,v)}} [\psi(X_{t \wedge \tau_n}, U_{t \wedge \tau_n})] \\ &= \mathbb{E}_{\mathbb{P}_{(x,u-2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x))}} [\psi(X_{t \wedge \tau_{n-1}}, U_{t \wedge \tau_{n-1}})] \\ &= \Gamma_{n-1}^\psi(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)). \end{aligned}$$

Using again the Feynmann–Kac formula, we then identify  $\Gamma_n^\psi$  as the solution to the Kolmogorov equation (2.18) with  $f_0 = \psi$  and  $q(t, x, u) = \Gamma_{n-1}^\psi(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x))$ .

Hence, given  $\Gamma_{n-1}^\psi$  continuous on  $\Sigma_T^-$ , Theorem 2.6 ensures that  $\Gamma_n^\psi(t, x, u)$  is smooth and the energy estimate (2.21) gives

$$\|\Gamma_n^\psi(t)\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \sigma^2 \|\nabla_u \Gamma_n^\psi\|_{L^2(Q_t)}^2 + \|\Gamma_n^\psi\|_{L^2(\Sigma_t^-)}^2 = \|\psi\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \|q\|_{L^2(\Sigma_t^+)}^2, \quad \forall t \in (0, T).$$

Since  $\|q\|_{L^2(\Sigma_t^+)}^2 = \|\Gamma_{n-1}^\psi\|_{L^2(\Sigma_t^-)}^2$ , this yields (2.19).  $\square$

**Corollary 2.7.** *Assume  $(H_{\text{Langevin}})$ . For all nonnegative  $\psi \in C_c^\infty(\mathcal{D} \times \mathbb{R}^d)$ ,  $\Gamma^\psi$  belongs to  $L^2((0, T) \times \mathcal{D}; H^1(\mathbb{R}^d))$  satisfies the energy equality:*

$$\|\Gamma^\psi(t)\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \sigma^2 \|\nabla_u \Gamma^\psi\|_{L^2(Q_t)}^2 = \|\psi\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2, \quad \forall t \in (0, T). \quad (2.23)$$

Furthermore,  $\Gamma^\psi(t)$  is solution in the sense of distributions of

$$\begin{cases} \partial_t \Gamma^\psi - (u \cdot \nabla_x \Gamma^\psi) - \frac{\sigma^2}{2} \Delta_u \Gamma^\psi = 0, & \text{on } Q_T, \\ \Gamma^\psi(0, x, u) = \psi(x, u), & \text{on } \mathcal{D} \times \mathbb{R}^d, \\ \Gamma^\psi(t, x, u) = \Gamma^\psi(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), & \text{on } \Sigma_T^+. \end{cases} \quad (2.24)$$

*Proof.* We first observe that since  $\psi|_{\partial \mathcal{D} \times \mathbb{R}^d} = 0$ ,

$$\Gamma_n^\psi(t, x, u) = \mathbb{E}_{\mathbb{P}_{t,x,u}} [\psi(X_{T \wedge \tau_n}, U_{T \wedge \tau_n})] = \mathbb{E}_{\mathbb{P}_{t,x,u}} [\psi(X_T, U_T) \mathbb{1}_{\{\tau_n \geq T\}}].$$

There exists a nonnegative function  $\beta \in L^2(\mathbb{R})$  such that  $\beta(|u|) = 1$  on the support of  $\psi$  and  $c\beta(|u|) \leq \psi \leq C\beta(|u|)$ , with  $C := \sup_{(x,u) \in \mathcal{D} \times \mathbb{R}^d} \psi(x, u)$ ,  $c := \inf_{(x,u) \in \mathcal{D} \times \mathbb{R}^d} \psi(x, u)$ . Then this initial boundary yields

$$c\mathbb{E}_{\mathbb{P}_{x,u}} [\beta(U_t) \mathbb{1}_{\{\tau_n \geq t\}}] \leq \Gamma_n^\psi(t, x, u) \leq C\mathbb{E}_{\mathbb{P}_{x,u}} [\beta(U_t) \mathbb{1}_{\{\tau_n \geq t\}}].$$

As  $\beta(|U_t|) = (G(\sigma t) * \beta)(|u|)$  where  $G$  denotes the heat kernel on  $\mathbb{R}^d$ , we obtain

$$c(G(\sigma t) * \beta)(|u|) \leq \Gamma_n^\psi(t, x, u) \leq C(G(\sigma t) * \beta)(|u|), \quad \text{on } Q_T. \quad (2.25)$$

Owing to the continuity of  $\Gamma_n^\psi$ , from the interior of  $Q_T$  to its boundary, (2.25) still holds true along  $\Sigma_T^\pm$ .

Let us now observe that, for a.e.  $(x, u) \in (\mathcal{D} \times \mathbb{R}^d) \cup (\Sigma \setminus \Sigma^0)$ ,  $\mathbb{P}_{(x,u)}$ -a.s.  $\tau_n$  grows to  $\infty$  as  $n$  increases, and then

$$\lim_{n \rightarrow +\infty} \Gamma_n^\psi(t, x, u) = \Gamma^\psi(t, x, u), \quad \text{for a.e. } (t, x, u) \in Q_T, \quad \lambda_\Sigma\text{-a.e. } (t, x, u) \in \Sigma_T \setminus \Sigma_T^0. \quad (2.26)$$

Indeed,

$$|\Gamma_n^\psi(t, x, u) - \Gamma^\psi(t, x, u)| = |\mathbb{E}_{\mathbb{P}_{x,u}} [\psi(X_t, U_t) \mathbb{1}_{\{\tau_n \leq t\}}]| \leq \|\psi\|_\infty \mathbb{P}_{x,u}(\tau_n \leq t).$$

In particular, (2.25) is also true for  $\Gamma^\psi$ . Moreover, by the Lebesgue Dominated Convergence Theorem,  $\Gamma_n^\psi$  converges to  $\Gamma^\psi$  in  $L^2(\mathcal{D} \times \mathbb{R}^d)$ .

We next deduce that the norms involving  $\Gamma_n^\psi$  in the left-hand side of (2.19) are finite for all  $t$ , uniformly in  $n$  (as the right-hand side of (2.19) is bounded uniformly in  $n$  by the Maxwellian bound (2.25)). Therefore, the estimate (2.19) is also true for  $\Gamma^\psi$  (see e.g. [8]), and  $\Gamma_n^\psi$  converges to  $\Gamma^\psi$  in  $L^2((0, T) \times \mathcal{D}; H^1(\mathbb{R}^d))$ .  $\square$

**Lemma 2.8** ( $L^p$ -control). *Given  $\psi \in C_c^\infty(\mathcal{D} \times \mathbb{R}^d)$  nonnegative, the kernels  $\Gamma_n^\psi$  and  $\Gamma_\psi$  considered in Proposition 2.5, satisfies for all  $p \in (1, +\infty)$*

$$\|\Gamma_n^\psi(t)\|_{L^p(\mathcal{D} \times \mathbb{R}^d)}^p + \|\Gamma_n^\psi\|_{L^p(\Sigma_t^-)}^p \leq \|\psi\|_{L^p(\mathcal{D} \times \mathbb{R}^d)}^p + \|\Gamma_{n-1}^\psi\|_{L^p(\Sigma_t^-)}^p, \quad \forall t \in (0, T), \quad (2.27)$$

$$\|\Gamma^\psi(t)\|_{L^p(\mathcal{D} \times \mathbb{R}^d)}^p \leq \|\psi\|_{L^p(\mathcal{D} \times \mathbb{R}^d)}^p, \quad \forall t \in (0, T). \quad (2.28)$$

*Proof.* Applying the estimate (2.22) to the solution to (2.18), it follows that for all  $t \in (0, T)$ ,

$$\|\Gamma_n^\psi(t)\|_{L^p(\mathcal{D} \times \mathbb{R}^d)}^p + \|\Gamma_n^\psi\|_{L^p(\Sigma_t^-)}^p \leq \|\psi\|_{L^p(\mathcal{D} \times \mathbb{R}^d)}^p + \|\Gamma_n^\psi\|_{L^p(\Sigma_t^+)}^p.$$

Since  $\|\Gamma_n^\psi\|_{L^p(\Sigma_t^+)}^p = \|\Gamma_{n-1}^\psi\|_{L^p(\Sigma_t^-)}^p$  we deduce (2.27).

Using the convergence of  $\Gamma_n^\psi$  to  $\Gamma^\psi$  and the uniform bounds (2.25) on  $\Sigma_T^-$ , we also deduce (2.28).  $\square$

### 2.3 On the boundary value problem (2.20)

In this section, we give the proof of Theorem 2.6. Consider the inputs  $(f_0, q)$  and assume that

$$H_{(f_0, q)}: \quad f_0 \in L^2(\mathcal{D} \times \mathbb{R}^d) \cap \mathcal{C}_b(\mathcal{D} \times \mathbb{R}^d) \text{ and } q \in L^2(\Sigma_T^+) \cap \mathcal{C}_b(\Sigma_T^+) \text{ are nonnegative functions.}$$

The main difficulty in the well-posedness of the boundary value problem (2.20) lies in the degeneracy of the diffusion operator and in the fact that we want to obtain the regularity of  $f$  along  $\Sigma_T \setminus \Sigma_T^+$ . Such problem has been addressed in Fichera [15] for second order differential operators of the form

$$\mathcal{L}(f)(z) = \text{Trace}(a(z)\nabla_z^2 f(z)) + (b(z) \cdot \nabla f(z)) + c(z)f(z) - h(z),$$

where  $a(z)$  is only assumed to be a weakly elliptic matrix; that is  $(\xi \cdot a(z)\xi) \geq 0$ , for all  $z \in \mathcal{U}$ ,  $\xi \in \mathbb{R}^N$ . Consider then the PDE

$$\mathcal{L}(f) = 0, \text{ on some smooth bounded open domain } \mathcal{U} \subset \mathbb{R}^N, \quad (2.29)$$

(in which belong the ultra-parabolic equations, among them the Langevin equation), submitted to some Dirichlet boundary condition. Denoting  $\nu(z)$  the unit outward normal vector to the boundary  $\partial\mathcal{U}$ ,  $\partial\mathcal{U}$  may be split into four parts: the so-called *non-characteristic part*  $\Sigma_3 := \{z \in \partial\mathcal{U}; (\nu(z) \cdot a(z)\nu(z)) > 0\}$ , the *relevant* part  $\Sigma_2 := \{z \in \partial\mathcal{U}/\Sigma_3; (b(z) \cdot \nu(z)) + \text{Trace}(a(z)\nabla_z \nu(z)) > 0\}$ , the *irrelevant* part  $\Sigma_1 := \{z \in \partial\mathcal{U}/\Sigma_3; (b(z) \cdot \nu(z)) + \text{Trace}(a(z)\nabla_z \nu(z)) < 0\}$  and the *sticking* part  $\Sigma_0 := \{z \in \partial\mathcal{D}/\Sigma_3; (b(z) \cdot \nu(z)) + \text{Trace}(a(z)\nabla_z \nu(z)) = 0\}$ . The term *relevant* refers to the boundary part where the boundary condition have to be specified; that is

$$f = g, \text{ on } \Sigma_{2,3} = \Sigma_2 \cup \Sigma_3. \quad (2.30)$$

The existence of solutions  $f$  in  $\mathcal{C}(\mathcal{D} \cup \Sigma_{2,3})$  to (2.29)–(2.30) has been studied by several authors, among them Kohn & Nirenberg [19], Oleřnik [25], Bony [3], and also Manfredini [23] in the context of ultra-parabolic equations endowing a full Dirichlet boundary condition along position–velocity domain (velocity space is assumed to be bounded). Stochastic interpretation of (2.29)–(2.30) has been studied in Strook & Varadhan [28], Freidlin [16], and Friedman [17]. However to the best of our knowledge, the regularity of  $f$  along  $\Sigma_1$  has not been considered outside a few works. We shall mention the *elliptic regularization method* introduced in Oleřnik & Radkevič [26] and Taira [29] which show the wellposedness of analytic solutions (on  $\Sigma_0 \cup \Sigma_1$ ) to the elliptic equation:

$$\begin{cases} \text{Trace}(a\nabla^2 f) + (b \cdot \nabla f) + cf - h = 0, & \text{on } \mathcal{U}, \\ f = 0, & \text{on } \Sigma_{2,3}, \end{cases} \quad (2.31)$$

under the particular assumption that the sets  $(\Sigma_i, i = 0, 1, 2, 3)$  are closed and that  $\overline{\Sigma_2 \cup \Sigma_3}$  and  $\overline{\Sigma_0 \cup \Sigma_1}$  are disjoint. Note that such assumption does not hold in the situation of kinetic equation. In that case, existence of weak solution is well known (see, e.g., Degond [12], Carrillo [9]). In particular, [9] treated the situation where (2.32) endows the specular boundary condition (1.3) and, following ideas of Degond & Mas-Gallic [13] established the existence of trace functions and Green identity related to the transport operator  $-\partial_t \psi - (u \cdot \nabla_x \psi)$ . As a preliminary result, let us recall a well-known existence results for (2.6),

**Proposition 2.9** (Carrillo [9], Theorem 2.2, Proposition 2.4 and Lemma 3.4). *Given two nonnegative functions  $f_0 \in L^2(\mathcal{D} \times \mathbb{R}^d)$  and  $q \in L^2(\Sigma_T)$ , there exists a unique nonnegative function  $f$  in  $\mathcal{C}([0, T]; L^2(\mathcal{D} \times \mathbb{R}^d) \cap L^2((0, T) \times \mathcal{D}; H^1(\mathbb{R}^d)))$  satisfying equation (2.20). Furthermore,  $f$  admits a nonnegative trace  $\gamma(f)$  along the boundary  $\Sigma_T$  in the sense that, for all  $t \in [0, T]$  and  $\psi \in \mathcal{C}_b^\infty(\overline{Q}_t)$ ,*

$$\begin{aligned} & \int_{Q_t} f(s, x, u) \left( \partial_s \psi - (u \cdot \nabla_x \psi) - \frac{\sigma^2}{2} \Delta_u \psi \right) (s, x, u) ds dx du \\ &= \int_{\mathcal{D} \times \mathbb{R}^d} [\psi(s, x, u) f(s, x, u)]_{s=0}^{s=t} dx du - \int_{\Sigma_t} (u \cdot n_{\mathcal{D}}(x)) \gamma(f)(s, x, u) \psi(s, x, u) d\lambda_{\Sigma_t}(s, x, u). \end{aligned} \quad (2.32)$$

In particular, for all  $t \in (0, T)$ ,

$$\|f(t)\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \sigma^2 \|\nabla_u f\|_{L^2(Q_t)}^2 + \|\gamma(f)\|_{L^2(\Sigma_t^-)}^2 = \|f_0\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \|q\|_{L^2(\Sigma_t^+)}^2. \quad (2.33)$$

If, in addition  $f_0 \in L^p(\mathcal{D} \times \mathbb{R}^d)$ ,  $q \in L^p(\Sigma_T^+)$ , then for all  $t \in (0, T)$ ,

$$\|f(t)\|_{L^p(\mathcal{D} \times \mathbb{R}^d)}^p + \|f\|_{L^p(\Sigma_t^-)}^p + \sigma^2 p(p-1) \|\nabla_u f\|_{L^p(Q_t)}^p \leq \|f_0\|_{L^p(\mathcal{D} \times \mathbb{R}^d)}^p + \|q\|_{L^p(\Sigma_t^+)}^p.$$

**Remark 2.10.** The link between the abstract Cauchy problem (2.20) and (2.32) should be seen as a consequence of the Green formula related to the transport operator  $\mathcal{T} = \partial_t + (u \cdot \nabla_x)$ . In particular, if  $f \in L^2((0, T) \times \mathcal{D}; H^1(\mathbb{R}^d))$  satisfies (2.32) then  $f$  is a solution, in the sense of distributions, to (2.20). Reciprocally, any distribution  $f \in L^2((0, T) \times \mathcal{D}; H^1(\mathbb{R}^d))$  solution to (2.20) verifies (2.32). For the sake of completeness, we clarify this point in Section 3.

Considering the solution  $f$  in  $\mathcal{C}([0, T]; L^2(\mathcal{D} \times \mathbb{R}^d) \cap L^2((0, T) \times \mathcal{D}; H^1(\mathbb{R}^d)))$  of (2.20), given by Proposition 2.9, we show its interior smoothness and its continuity up to and along  $\Sigma_T \setminus \Sigma_T^0$ .

**Interior regularity.** For the interior regularity of  $f$ , let us recall the Schauder estimate for weak solution of ultra-parabolic equations due to Manfredini [23] and Di Francesco & Polidoro [14].

**Proposition 2.11** ([23] and [14]). Assume  $(H_{(f_0, q)})$ . Let  $f$  be the weak solution to (2.32) with inputs  $(f_0, q)$  given in Proposition 2.9. Then, for all  $t \in (0, T]$ ,  $(x_0, u_0) \in \mathcal{D} \times \mathbb{R}^d$  and  $r > 0$  such that  $B(x_0, r) \subset \mathcal{D}$ ,  $f$  belongs to  $\mathcal{C}^{1,1,2}((0, T] \times B(x_0, r) \times \mathbb{R}^d)$  and, for all  $\alpha \in (0, 1)$ , there exists  $C_T > 0$  such that

$$\|f(t, \cdot)\|_{\alpha, t, B((x_0, u_0), r)} \leq C_T \|f_0\|_{L^\infty(B((x_0, u_0), r))},$$

where  $\|\cdot\|_{\alpha, t, B((x_0, u_0), r)}$  defined by

$$\begin{aligned} & \|\psi\|_{\alpha, t, B((x_0, u_0), r)} \\ &= \sup_{B((x_0, u_0), r)} |\psi(t, \cdot)| + \sqrt{t} \sup_{B((x_0, u_0), r)} |\nabla_u \psi(t, \cdot)| + t \sup_{B((x_0, u_0), r)} (|\mathcal{T}(\psi)(t, \cdot)| + |\nabla_u^2 \psi(t, \cdot)|) \\ &+ t^{\frac{2+\alpha}{2}} \sup_{(x, u), (y, v) \in B((x_0, u_0), r)} \left( \frac{|\psi(t, x, u) - \psi(t, y, v)|}{|(x, u) - (y, v)|^\alpha} + \frac{|\nabla_u \psi(t, x, u) - \nabla_u \psi(t, y, v)|}{|(x, u) - (y, v)|^\alpha} \right) \\ &+ t^{\frac{2+\alpha}{2}} \sup_{(x, u), (y, v) \in B((x_0, u_0), r)} \left( \frac{|\mathcal{T}(\psi)(t, x, u) - \mathcal{T}(\psi)(t, y, v)|}{|(x, u) - (y, v)|^\alpha} + \frac{|\nabla_u^2 \psi(t, x, u) - \nabla_u^2 \psi(t, y, v)|}{|(x, u) - (y, v)|^\alpha} \right). \end{aligned}$$

with  $\mathcal{T} = \partial_t + (u \cdot \nabla_x)$ .

In addition, we have the following  $L^\infty$ -estimate:

**Lemma 2.12** ([9], Lemma 3.4). Let  $f$  be the weak solution to (2.32) with inputs  $(f_0, q)$  as in Proposition 2.11. Then  $f$  satisfies

$$\left( \|\gamma(f)\|_{L^\infty(\Sigma_t^-)} \vee \|f(t)\|_{L^\infty(\mathcal{D} \times \mathbb{R}^d)} \right) \leq \|q\|_{L^\infty(\Sigma_t^+)} + \|f_0\|_{L^\infty(\mathcal{D} \times \mathbb{R}^d)}, \quad \forall t \in (0, T].$$

Combining Proposition 2.11 and Lemma 2.12 enables us to conclude that  $f$  belongs to  $\mathcal{C}^{1,1,2}(Q_T)$ .

At this point it remains to check that  $f$  is continuous up to and along  $\Sigma_T^\pm$ . This is the aim of the next two paragraphs.

**Continuity up to  $\Sigma_T^+$ .**

**Proposition 2.13.** Assume  $(H_{\text{Langevin}})$ -(ii) and  $(H_{(f_0, q)})$ . Let  $f \in \mathcal{C}^{1,1,2}(Q_T)$  be solution to (2.32). Then  $f$  is continuous up to  $\Sigma_T^+$ .

*Proof.* To show the continuity of  $f$  up to the boundary  $\Sigma_T^+$ , we follow the classical construction of local barrier functions to all points  $(t_0, x_0), u_0$  of  $\Sigma_T^+$  (see e.g. [18], p. 104–106, 118–120). Let  $(t_0, x_0, u_0) \in \Sigma_T^+$ . Since  $q$  is continuous in  $\Sigma_T^+$ , we can assume that for any  $\epsilon > 0$ , there exists a neighborhood  $\mathcal{O}_{t_0, x_0, u_0}$  of  $(t_0, x_0, u_0)$  such that

$$q(t_0, x_0, u_0) - \epsilon \leq q(t, x, u) \leq q(t_0, x_0, u_0) + \epsilon, \quad \forall (t, x, u) \in \mathcal{O}_{t_0, x_0, u_0} \cap \Sigma_T^+.$$

In addition, since  $(u_0 \cdot n_{\mathcal{D}}(x_0))$  is positive, by reducing  $\mathcal{O}_{t_0, x_0, u_0}$ , we can still assume that  $\varrho \in \mathcal{C}^2(\overline{\mathcal{O}_{t_0, x_0, u_0}})$  and that  $(u \cdot n_{\mathcal{D}}(x))\eta$  on for all  $(t, x, u) \in \mathcal{O}_{t_0, x_0, u_0}$ , for some positive  $\eta$ <sup>1</sup>. Consequently, by setting  $\varrho(x) = -\sigma(x)$ , where  $\sigma$  is the signed distance to  $\partial\mathcal{D}$  given in (2.2), and setting

$$L := \partial_t - (u \cdot \nabla_x) - \frac{\sigma^2}{2} \Delta_u,$$

we observe that, for all  $(t, x, u) \in \mathcal{O}_{t_0, x_0, u_0}$ ,

$$L(\varrho)(t, x, u) = -(u \cdot \nabla_x \varrho(x)) = (u \cdot n_{\mathcal{D}}(x)) > 0. \quad (2.34)$$

Finally, reducing again  $\mathcal{O}_{t_0, x_0, u_0}$ , we can assume that  $\mathcal{O}_{t_0, x_0, u_0}$  has the form  $(t_0 - \delta, t_0 + \delta) \times B_{x_0}(\delta') \times B_{u_0}(\delta')$  (where  $B_{x_0}(\delta')$  [resp.  $B_{u_0}(\delta')$ ] the ball centered in  $x_0$  [resp.  $u_0$ ] of radius  $\delta'$ ) for some positive constants  $\delta, \delta' > 0$  such that  $0 \leq t_0 - \delta < t_0 + \delta \leq T$ .

Therefore, we can construct the barrier functions related to  $(t_0, x_0, u_0) \in \Sigma_T^+$  with

$$\begin{aligned} \overline{\omega}_\epsilon(t, x, u) &= q(t_0, x_0, u_0) + \epsilon + k_\epsilon \psi_{x_0}(x) + K_\epsilon \varrho(x), \\ \underline{\omega}_\epsilon(t, x, u) &= q(t_0, x_0, u_0) - \epsilon - k_\epsilon \psi_{x_0}(x) - K_\epsilon \varrho(x). \end{aligned} \quad (2.35)$$

where  $\psi_{x_0}(x) = (x - x_0)^2$  and where the parameters  $k_\epsilon, K_\epsilon \in \mathbb{R}^+$  are chosen large enough so that, for  $M^+$  a upper-bound of  $f$  on  $\partial\mathcal{O}_{t_0, x_0, u_0} \cap Q_T$ , we have

$$\begin{aligned} k_\epsilon \inf_{\mathcal{O}_{t_0, x_0, u_0} \cap Q_T} L(\psi_{x_0}) + K_\epsilon \inf_{\mathcal{O}_{t_0, x_0, u_0} \cap Q_T} L(\varrho) &\geq 0, \\ k_\epsilon \inf_{\partial\mathcal{O}_{t_0, x_0, u_0} \cap Q_T} \psi_{x_0} + K_\epsilon \inf_{\partial\mathcal{O}_{t_0, x_0, u_0} \cap Q_T} \varrho &\geq \sup(M^+ - (q(t_0, x_0, u_0) + \epsilon), q(t_0, x_0, u_0) - \epsilon). \end{aligned}$$

For example, setting  $\eta := \inf_{\mathcal{O}_{t_0, x_0, u_0} \cap Q_T} (u \cdot n_{\mathcal{D}}(x))$ , if  $\psi_{x_0}(t, x, u) = (x - x_0)^2$ , then one can take  $k_\epsilon$  and  $K_\epsilon$  as

$$-(\delta')^2 k_\epsilon + K_\epsilon \eta = 0, \quad k_\epsilon (\delta')^2 \left( = k_\epsilon \inf_{\partial\mathcal{O}_{t_0, x_0, u_0} \cap Q_T} \psi_{x_0} \right) = \sup(M^+ - q(t_0, x_0, u_0), q(t_0, x_0, u_0)).$$

Therefore,  $\overline{\omega}_\epsilon$  and  $\underline{\omega}_\epsilon$  satisfy the properties

$$(P) \quad \begin{cases} (a) \ \overline{\omega}_\epsilon(t, x, u) \geq q(t, x, u) \geq \underline{\omega}_\epsilon(t, x, u) \text{ for all } (t, x, u) \in \mathcal{O}_{t_0, x_0, u_0} \cap (0, T) \times \partial\mathcal{D} \times \mathbb{R}^d, \\ (b) \ L(\overline{\omega}_\epsilon) \geq 0 \geq L(\underline{\omega}_\epsilon) \text{ for all } (t, x, u) \in \mathcal{O}_{t_0, x_0, u_0} \cap Q_T, \\ (c) \ \overline{\omega}_\epsilon(t, x, u) \geq M^+, \text{ and } \underline{\omega}_\epsilon(t, x, u) \leq M^-, \text{ for all } (t, x, u) \in \partial\mathcal{O}_{t_0, x_0, u_0} \cap Q_T, \\ (d) \ \lim_{\epsilon \rightarrow 0^+} \overline{\omega}_\epsilon(t_0, x_0, u_0) = \lim_{\epsilon \rightarrow 0^+} \underline{\omega}_\epsilon(t_0, x_0, u_0) = q(t_0, x_0, u_0). \end{cases}$$

We shall now prove that, for  $f$  the solution to (2.32),  $\underline{\omega}_\epsilon \leq f \leq \overline{\omega}_\epsilon$  on  $\mathcal{O}_{t_0, x_0, u_0} \cap Q_T$ . Owing to the property  $P(d)$ , this is enough to conclude that  $f(t, x, u)$  tends to  $q(t_0, x_0, u_0)$  as  $(t, x, u)$  tends to  $(t_0, x_0, u_0)$ , for all  $(t_0, x_0, u_0)$  of  $\Sigma_T^+$ . To show local comparisons between  $\overline{\omega}_\epsilon$  and  $f$ , let us consider the positive part,  $(f - \overline{\omega}_\epsilon)^+$ , of  $f - \overline{\omega}_\epsilon$ . We recall that The derivative of the positive (as well as the negative) part of a Sobolev function is well-defined (see e.g. Tartar [30] and Theorem A.2 in the Appendix). In addition, let  $\eta_0$  denote some nonnegative cut-off function defined in a neighborhood of  $(t_0, x_0, u_0)$  such that  $\eta_0(t, x, u) = 0$  for all  $(t, x, u) \in \partial\mathcal{O}_{t_0, x_0, u_0}$ , and let  $\beta$  be a real parameter that we will specify later. where the function  $\Delta_u |(f - \overline{\omega}_\epsilon)^+|^2$  is well defined a.e. on  $Q_T$  since, using Theorem A.2, one can check that  $\Delta_u |(f - \overline{\omega}_\epsilon)^+|^2 = 2 \nabla_u \cdot ((f - \overline{\omega}_\epsilon)^+ \nabla_u (f - \overline{\omega}_\epsilon)) = 2((f - \overline{\omega}_\epsilon)^+ \Delta_u (f - \overline{\omega}_\epsilon)) + 2 |\nabla_u (f - \overline{\omega}_\epsilon)|^2 1_{\{f > \overline{\omega}_\epsilon\}}$ . We shall observe that

$$\begin{aligned} &L(\eta_0 \exp\{\beta t\} |(f - \overline{\omega}_\epsilon)^+|^2) \\ &= \eta_0 \exp\{\beta t\} L(|(f - \overline{\omega}_\epsilon)^+|^2) + |(f - \overline{\omega}_\epsilon)^+|^2 L(\eta_0 \exp\{\beta t\}) + \sigma^2 \exp\{\beta t\} \left( \nabla_u \eta_0 \cdot |(f - \overline{\omega}_\epsilon)^+|^2 \right) \end{aligned}$$

<sup>1</sup>For instance, assuming that  $\varrho \in \mathcal{C}^2$  for all  $x$  such that  $d(x, \partial\mathcal{D}) \leq \mu$ , by setting  $C := (u_0 \cdot n_{\mathcal{D}}(x_0))$  – which is positive – we can choose  $0 < \delta' < \mu$  so that  $\delta' \left( \sup_{d(x, \partial\mathcal{D}) \leq \mu} |n_{\mathcal{D}}(x)| + \sup_{d(x, \partial\mathcal{D}) \leq \mu} |\nabla_x n_{\mathcal{D}}(x)| + \delta' + |u_0| \right) < C$ . Therefore, we have  $(u \cdot n_{\mathcal{D}}(x)) \geq (u_0 \cdot n_{\mathcal{D}}(x_0)) - |(u_0 \cdot n_{\mathcal{D}}(x_0)) - (u \cdot n_{\mathcal{D}}(x))| \geq C - \delta' \sup_{d(x, \partial\mathcal{D}) \leq \mu} |n_{\mathcal{D}}(x)| + \delta' \left( \sup_{d(x, \partial\mathcal{D}) \leq \mu} |\nabla_x n_{\mathcal{D}}(x)| + \delta' + |u_0| \right) > 0$ .

The property (P)(b) ensures that

$$L(|(f - \bar{\omega}_\epsilon)^+|^2) = L(f - \bar{\omega}_\epsilon)(f - \bar{\omega}_\epsilon)^+ - \frac{\sigma^2}{2} |\nabla_u(f - \bar{\omega}_\epsilon)|^2 1_{\{f > \bar{\omega}_\epsilon\}} \leq -\frac{\sigma^2}{2} |\nabla_u(f - \bar{\omega}_\epsilon)|^2 1_{\{f > \bar{\omega}_\epsilon\}},$$

it follows that

$$\begin{aligned} & L(\eta_0 \exp\{\beta t\} |(f - \bar{\omega}_\epsilon)^+|^2) \\ & \leq -\frac{\sigma^2 \eta_0 \exp\{\beta t\}}{2} |\nabla_u(f - \bar{\omega}_\epsilon)|^2 1_{\{f > \bar{\omega}_\epsilon\}} + (L(\eta_0) + \beta \eta_0) \exp\{\beta t\} |(f - \bar{\omega}_\epsilon)^+|^2 - \sigma^2 \exp\{\beta t\} \left( \nabla_u \eta_0 \cdot \nabla_u |(f - \bar{\omega}_\epsilon)^+|^2 \right) \\ & \leq (L(\eta_0) + \beta \eta_0) \exp\{\beta t\} |(f - \bar{\omega}_\epsilon)^+|^2 - \sigma^2 \exp\{\beta t\} \left( \nabla_u \eta_0 \cdot \nabla_u |(f - \bar{\omega}_\epsilon)^+|^2 \right). \end{aligned}$$

Integrating the above over  $\mathcal{O}_{t_0, x_0, u_0} \cap Q_{T_0}$  and since  $\eta_0 = 0$  on  $\partial \mathcal{O}_{t_0, x_0, u_0}$ , integration by parts yields

$$\begin{aligned} & - \int_{\Sigma_T \cap \mathcal{O}_{t_0, x_0, u_0}} (u \cdot n_{\mathcal{D}}(x)) \frac{\eta_0(t, x, u)}{2} \exp\{\beta t\} |(\gamma(f) - \bar{\omega}_\epsilon)^+(t, x, u)|^2 d\lambda_\Sigma(t, x, u) \\ & \leq \int_{Q_T \cap \mathcal{O}_{t_0, x_0, u_0}} \left( \left( \frac{1}{2}L - \frac{1}{4}\Delta_u \right) (\eta_0) + \beta \eta_0 \right) \exp\{\beta t\} |(f - \bar{\omega}_\epsilon)^+|^2, \end{aligned}$$

or equivalently, since  $\eta_0 = 0$  on  $\Sigma_T^-$ ,

$$\begin{aligned} & - \int_{\Sigma_T^+ \cap \mathcal{O}_{t_0, x_0, u_0}} (u \cdot n_{\mathcal{D}}(x)) \frac{\eta_0(t, x, u)}{2} \exp\{\beta t\} |(\gamma(f) - \bar{\omega}_\epsilon)^+(t, x, u)|^2 d\lambda_\Sigma(t, x, u) \\ & \leq \int_{Q_T \cap \mathcal{O}_{t_0, x_0, u_0}} \left( \left( \frac{1}{2}L - \frac{1}{4}\Delta_u \right) (\eta_0) + \beta \eta_0 \right) \exp\{\beta t\} |(f - \bar{\omega}_\epsilon)^+|^2 \end{aligned}$$

Since and according to (P)(a) and (P)(c), the integral along  $\Sigma_T^+$  and along  $\partial \mathcal{O}_{t_0, x_0, u_0}$  are nonnegative. Choosing  $\eta$  such that  $|\nabla_x \eta_0(t, x, u)| + |\Delta_u \eta_0(t, x, u)| \leq R\eta(t, x, u)$ , for some constant  $R > 0$ , and  $\beta$  small enough so that

$$\left( \frac{1}{2}L - \frac{1}{4}\Delta_u \right) (\eta_0) + \beta \eta_0 \leq 0$$

ensures that  $f \leq \bar{\omega}_\epsilon$  on  $\mathcal{O}_{t_0, x_0, u_0}$ . Similar arguments entail that  $\underline{\omega}_\epsilon \leq f$ . This ends the proof.  $\square$

**Continuity up to and along  $\Sigma_T^-$ .**

**Proposition 2.14.** *Let  $f$  be the weak solution to (2.20) with  $f_0 \in L^2(\mathcal{D} \times \mathbb{R}^d) \cap \mathcal{C}_b(\mathcal{D} \times \mathbb{R}^d)$  and  $q \in L^2(\Sigma_T^-) \cap \mathcal{C}_b(\Sigma_T^-)$ . Then  $f$  is continuous along and up to  $\Sigma_T^-$ .*

*Proof.* The proof relies on a probabilistic interpretation. For  $(y, v) \in \mathbb{R}^{2d}$  fixed, let  $(x_t^{(y, v)}, u_t^{(y, v)})_{t \in [0, T]}$  satisfies

$$\begin{cases} x_t^{(y, v)} = y + \int_0^t u_s^{(y, v)} ds, \\ u_t^{(y, v)} = v + W_t, \end{cases}$$

for  $W$  a  $\mathbb{R}^d$ -valued Brownian motion defined on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Set  $\beta_\delta^{(y, v)} := \inf\{t > 0; d(x_t^{(y, v)}, \partial \mathcal{D}) \leq \delta\}$ . Since  $f$  is smooth in the interior of  $Q_T$  and satisfies (2.32), the Feynman-Kac formula yields:

$$f(t, (y, v)) = \mathbb{E}_{\mathbb{P}} \left[ f_0(x_t^{(y, v)}, u_t^{(y, v)}) \mathbb{1}_{\{t \leq \beta_\delta^{(y, v)}\}} \right] + \mathbb{E}_{\mathbb{P}} \left[ q(\beta_\delta^{(y, v)}, x_{\beta_\delta^{(y, v)}}^{(y, v)}, u_{\beta_\delta^{(y, v)}}^{(y, v)}) \mathbb{1}_{\{t > \beta_\delta^{(y, v)}\}} \right].$$

for  $(y, v) \in \mathcal{D} \times \mathbb{R}^d$ . Then, since  $\mathbb{P}$ -a.s.,  $\beta_\delta^{(y, v)}$  tends to  $\tau^{(y, v)} := \{t > 0; x_t^{(y, v)} \notin \bar{\mathcal{D}}\}$  as  $\delta$  tends to 0, and thanks to Proposition 2.13, one obtains the formula

$$f(t, y, v) = \mathbb{E}_{\mathbb{P}} \left[ f_0(x_{t \wedge \tau^{(y, v)}}^{(y, v)}, u_{t \wedge \tau^{(y, v)}}^{(y, v)}) \mathbb{1}_{\{t \leq \tau^{(y, v)}\}} \right] + \mathbb{E}_{\mathbb{P}} \left[ q(\tau^{(y, v)}, x_{\tau^{(y, v)}}^{(y, v)}, u_{\tau^{(y, v)}}^{(y, v)}) \mathbb{1}_{\{t > \tau^{(y, v)}\}} \right].$$

The continuity of  $f$  up to  $\Sigma_T^-$  will follow from the continuity of  $(y, v) \mapsto (\tau^{(y,v)}, x_t^{(y,v)}, u_t^{(y,v)})$ .  $\mathbb{P}$ -almost surely, the flow  $(y, v) \mapsto (x_t^{(y,v)}, u_t^{(y,v)})$  is continuous on  $\mathbb{R}^d$  for all  $t \geq 0$ , and  $(y, v) \mapsto \tau^{(y,v)}$  is continuous on  $\Sigma_T^-$ . To prove the later, we replicate the general proof of the continuity of exit time related to a flow of continuous processes given in Proposition 6.3 in Darling & Pardoux [11]. Replicating the argument of the authors, we can show that, for all  $(x_m, u_m) \in \mathcal{D} \times \mathbb{R}^d$  such that  $\lim_{m \rightarrow +\infty} (x_m, u_m) = (x, u)$ ,

$$\limsup_{m \rightarrow +\infty} \tau^{(x_m, u_m)} \leq \tau^{(x, u)}.$$

Next, it is sufficient to check that

$$\tau^{(x, u)} \leq \liminf_{m \rightarrow +\infty} \tau^{(x_m, u_m)}.$$

Following [11], we may observe that, as in the proof of Theorem 2.1, for a.e.  $(x, u) \in \mathcal{D} \times \mathbb{R}^d \cup \Sigma^-$ , the paths  $t \mapsto (x_t^{(x, u)}, u_t^{(x, u)})$  never hits  $\Sigma^0$ , and  $\mathbb{P}$ -a.s.  $(x_t^{x, u}, u_t^{x, u}, t \in [0, \liminf_{m \rightarrow +\infty} \tau^{(x_m, u_m)}])$  reaches the set

$$\overline{\{(x_{\tau^{(x_m, u_m)}}^{(x_m, u_m)}, u_{\tau^{(x_m, u_m)}}^{(x_m, u_m)}); m \in \mathbb{N}\}} \subset \Sigma^+.$$

Since  $\tau^{(x, u)} \leq \inf\{t > 0; (x_t^{(x, u)}, u_t^{(x, u)}) \in \Sigma^+\}$ , we deduce that  $\tau^{(x, u)} \in [0, \liminf_{m \rightarrow +\infty} \tau^{(x_m, u_m)}]$ .  $\square$

This ends the proof of Theorem 2.6.

### 3 The time-marginal densities of the solution of (1.1)

In this section, we construct a probability density function satisfying (in the sense of distribution):

$$\partial_t \rho + (u \cdot \nabla_x \rho) + (B[\cdot; \rho] \cdot \nabla_u \rho) - \frac{\sigma^2}{2} \Delta_u \rho = 0, \text{ on } (0, T) \times \mathcal{D} \times \mathbb{R}^d, \quad (3.1a)$$

$$\rho(0, x, u) = \rho_0(x, u), \text{ on } \mathcal{D} \times \mathbb{R}^d, \quad (3.1b)$$

$$\gamma(\rho)(t, x, u) = \gamma(\rho)(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), \text{ on } (0, T) \times \partial\mathcal{D} \times \mathbb{R}^d, \quad (3.1c)$$

where  $\gamma(\rho)$  stands for the trace of  $\rho$  in the sense of Definition 1.1, and  $B$  is defined as in (1.2). Clearly (3.1) corresponds to the governing equation of the law of  $(X_t, U_t)$  solution to (1.1), in particular (3.1c) corresponds to the effect of the confinement component  $(K_t)$ . Throughout this section, we refer to equation (3.1) as the conditional McKean-Vlasov-Fokker-Planck Equation. Furthermore, for notation convenience, we denote by  $\mathcal{T}$  the transport operator in (3.1a), namely for all test function on  $Q_T$ ,

$$\mathcal{T}(\psi) = \partial_t \psi + (u \cdot \nabla_x \psi). \quad (3.2)$$

As mentioned in Section 2.3, the well-posedness of the linear Vlasov-Fokker-Planck equation and the related trace problem has been well studied in the literature of kinetic equation (we particularly refer to Degond [12], Degond and Mas Gallic [13] and Carrillo [9]).

For the study of the conditional McKean-Vlasov-Fokker-Planck equation (3.1), the two main difficulties are in the fractional form of  $B[\cdot; \rho]$ , and in the verification of the properties (1.5a) and (1.5b) for the trace (see Definition 1.1). For this purpose, starting from the assumption  $(H_{\text{VFP}})$ -(iii), we exhibit the existence of Maxwellian upper and lower bounds for the solution of (3.1) of the following form.

**Definition 3.1.** For given  $a \in \mathbb{R}$ ,  $\mu > 0$ ,  $P_0 \in L^1(\mathbb{R}^d)$ , such that  $P_0 \geq 0$  on  $\mathbb{R}^d$ , a Maxwellian distribution with parameters  $(a, \mu, P_0)$  is a function  $P : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}^+$  such that

$$P(t, u) = \exp\{at\} [m(t, u)]^\mu, \quad (3.3)$$

where  $m : \mathbb{R}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}$  is defined by  $m(t, u) = (G(\sigma^2 t) * P_0^{\frac{1}{\mu}})(u)$ , with  $G(t, u) = \left(\frac{1}{2\pi t}\right)^{\frac{d}{2}} \exp\left\{-\frac{|u|^2}{2t}\right\}$ .

**Remark 3.2.** Let  $p$  be a Maxwellian distribution with parameters  $(a, \mu, p_0)$ . If  $p_0(u) = p_0(|u|)$  then, for all vector  $\vec{n} \in \mathbb{R}^d$  such that  $\|\text{vecn}\| = 1$ , the Maxwellian distribution satisfies

$$p(t, u - 2(u \cdot \vec{n})\vec{n}) = p(t, u), \text{ for a.e. } (t, u) \in (0, +\infty) \times \mathbb{R}^d.$$

This section is devoted to the proof of the following existence result.

**Theorem 3.3.** *Under  $(H_{\text{VFP}})$ , there exists a function  $\rho \in V_1(\omega, Q_T)$ , and there exist  $\gamma^+(\rho)$ ,  $\gamma^-(\rho)$  defined on  $\Sigma_T^+$  and  $\Sigma_T^-$  respectively, with  $\gamma^\pm(\rho) \in L^2(\omega, \Sigma_T^\pm)$ , such that, for all  $t \in (0, T]$ , for all  $\psi \in \mathcal{C}_c^\infty(\overline{Q_t})$ , it holds*

$$\begin{aligned} & \int_{Q_t} \left( \rho \mathcal{T}(\psi) + \psi (B[\cdot; \rho] \cdot \nabla_u \rho) + \frac{\sigma^2}{2} (\nabla_u \psi \cdot \nabla_u \rho) \right) (s, x, u) ds dx du \\ &= \int_{\mathcal{D} \times \mathbb{R}^d} \rho(t, x, u) \psi(t, x, u) dx du - \int_{\mathcal{D} \times \mathbb{R}^d} \rho_0(x, u) \psi(0, x, u) dx du \\ &+ \int_{\Sigma_t^+} (u \cdot n_{\mathcal{D}}(x)) \gamma^+(\rho)(s, x, u) \psi(s, x, u) d\lambda_\Sigma(s, x, u) \\ &+ \int_{\Sigma_t^-} (u \cdot n_{\mathcal{D}}(x)) \gamma^-(\rho)(s, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)) \psi(s, x, u) d\lambda_\Sigma(s, x, u). \end{aligned} \quad (3.4)$$

In addition, there exist a couple of Maxwellian distributions  $(\overline{P}, \underline{P})$  such that

$$\begin{aligned} \underline{P} &\leq \rho \leq \overline{P}, \text{ a.e. on } Q_T, \\ \underline{P} &\leq \gamma^\pm(\rho) \leq \overline{P}, \text{ } \lambda_\Sigma\text{-a.e. on } \Sigma_T^\pm, \end{aligned} \quad (3.5)$$

$\overline{P}$  and  $\underline{P}$  satisfy the specular boundary condition (3.1c), and for all  $t \in (0, T]$ ,

$$\int_{\mathbb{R}^d} (1 + |u|) \omega(u) |\overline{P}(t, u)|^2 du < +\infty, \quad (3.6a)$$

$$\int_{\mathbb{R}^d} \underline{P}(t, u) du > 0. \quad (3.6b)$$

The main steps of the proof of Theorem 3.3 are the following: first, we consider a linear version of equation (3.1) where a Dirichlet condition is imposed on  $\Sigma_T^-$ , and where the drift coefficient is given in  $L^\infty((0, T) \times \mathcal{D}; \mathbb{R}^d)$ . Under  $(H_{\text{VFP}})$ -(ii) and  $(H_{\text{VFP}})$ -(iii), the problem is well-posed in  $V_1(\omega, Q_T)$  (see Lemma 3.7). Then we show the existence of Maxwellian bounds satisfying the requirements (3.6) (see Proposition 3.8). Next, by means of fixed point methods relying on the bounds of the solution of the linear equation, we successively introduce the specular boundary condition (see Proposition 3.13) and the nonlinear term  $B[\cdot; \rho]$  (see Proposition 3.14) in the equations. As a preliminary step, we highlight in the next subsection some meaningful properties on the transport operator  $\mathcal{T}$  and the Green identity related to the Vlasov–Fokker–Planck equation in the weighted spaces  $V_1(\omega, Q_T)$ .

The Maxwellian distributions  $(\overline{P}, \underline{P})$  are identified as super-solution and sub-solution for (3.1), and enable to generate an explicit upper and lower limit for the solution, starting from well-suited initial bounds.

The weight  $\omega(u)$  defined in (1.8) is useful here to preserve the probabilistic interpretation of (3.1) while working in  $L^2$ -space. Later it also allows a fixed point argument. Let us remark the following properties:

**Lemma 3.4.** *For all  $u$  and  $u'$  in  $\mathbb{R}^d$ ,*

$$\begin{aligned} (i) \quad & \omega(u + u') \leq 2^{\frac{\alpha}{2}} (\omega(u) + \omega(u')) \\ (ii) \quad & (u \cdot \nabla_u \omega(u)) \geq 0, \text{ and } |\nabla_u \omega(u)| \leq \alpha \omega(u), \\ & \left| \left( u \cdot \sqrt{\omega(u)} \right) \right| \leq \frac{\alpha}{4} \sqrt{\omega(u)}, \\ & \left| \nabla_u \sqrt{\omega(u)} \right| \leq \frac{\alpha}{2} \sqrt{\omega(u)}, \\ & \Delta_u \omega(u) \leq \left( 2\alpha \left( \frac{\alpha}{2} - 1 \right) + \alpha d \right) \omega(u) \\ (iii) \quad & \int_{\mathbb{R}^d} \frac{du}{\omega(u)} < +\infty. \end{aligned}$$

*Proof.* The assertions (i) and (ii) are directly deduced from the calculations:

$$\begin{aligned}
(1 + |u + u'|^2)^{\frac{\alpha}{2}} &\leq (1 + 2|u|^2 + 2|u'|^2)^{\frac{\alpha}{2}}, \\
(u \cdot \nabla_u \omega(u)) &= \frac{\alpha}{2} |u|^2 (1 + |u|^2)^{\frac{\alpha}{2}-1}, \\
|\nabla_u \omega| &= \alpha |u| (1 + |u|^2)^{\frac{\alpha}{2}-1} \leq \alpha \omega(u), \\
|\nabla_u \sqrt{\omega(u)}| &= \frac{\alpha}{2} |u| (1 + |u|^2)^{\frac{\alpha}{4}-1}, \\
\Delta_u \omega(u) &= \alpha d (1 + |u|^2)^{\frac{\alpha}{2}-1} + 2\alpha \left( \frac{\alpha}{2} - 1 \right) |u|^2 (1 + |u|^2)^{\frac{\alpha}{2}-2}.
\end{aligned}$$

For the assertion (iii), by a change of variable in the polar coordinates, we have

$$\int_{\mathbb{R}^d} \frac{du}{\omega(u)} = C \int_{\mathbb{R}^+} (1 + r^2)^{-\frac{\alpha}{2}} r^{d-1} dr = |S_{d-1}| \int_{\mathbb{R}^+} (1 + |r|^2)^{-\frac{\alpha+d-1}{2}} dr$$

where  $|S_{d-1}|$  is the Lebesgue measure of the unit sphere of  $\mathbb{R}^d$ . Since the right member is finite for  $\alpha > d$ , we have (iii).  $\square$

Owing to Theorem 3.3, the function

$$\gamma(\rho)(t, x, u) = \begin{cases} \gamma^+(\rho)(t, x, u) & \text{on } \Sigma_T^+, \\ \gamma^-(\rho)(t, x, u) & \text{on } \Sigma_T^-, \end{cases}$$

is the trace of  $\rho$  in the sense of Definition 1.1. In particular, thanks to (3.6), the integrability and positivity requirement are then shifted to the Maxwellian bounds and consequently to the initial bounds given by  $(H_{\text{VFP}})$ . Using Lemma (3.4)-(iii), the upper bound for  $\gamma(\rho)$  in (3.5) and the property (3.6a) yield that

$$\begin{aligned}
\int_{\Sigma_T} |(u \cdot n_{\mathcal{D}}(x))| \gamma(\rho)(t, x, u) d\lambda_{\Sigma_T}(t, x, u) &= 2 \int_{\Sigma_T^+} |(u \cdot n_{\mathcal{D}}(x))| \gamma(\rho)^+(t, x, u) d\lambda_{\Sigma_T}(t, x, u) \\
&\leq C_1 \int_{\mathbb{R}^d} |u| |\overline{P}|(u) du \leq C_2 \|\overline{P}\|_{L^2(\omega, \Sigma_T)} < +\infty,
\end{aligned}$$

where  $C_1$  and  $C_2$  are constants depending only on  $\omega$  and  $\partial\mathcal{D}$ .

### 3.1 On the linear Vlasov-Fokker-Planck equation

In this section, we set the framework for the existence proof of Theorem 3.3 by recalling the spaces and the existence framework of the linear Vlasov-Fokker-Planck equation.

First we give some properties of the operator  $\mathcal{T}$  given in (3.2), that was initially stated in [9], inspired from ideas in [13] and [12]. For all  $t \in (0, T]$ , consider the space

$$\mathcal{Y}(Q_t) = \{\phi \in \mathcal{H}(Q_t) ; \mathcal{T}(\phi) \in \mathcal{H}'(Q_t)\},$$

equipped with the norm

$$\|\phi\|_{\mathcal{Y}(Q_t)}^2 = \|\phi\|_{\mathcal{H}(Q_t)}^2 + \|\mathcal{T}(\phi)\|_{\mathcal{H}'(Q_t)}^2,$$

and  $\tilde{\mathcal{Y}}(Q_t)$ , the subset of all elements of  $\mathcal{C}_c^\infty(\overline{Q_t})$ , vanishing at the neighborhood of the boundaries  $\{0\} \times \partial\mathcal{D} \times \mathbb{R}^d$ ,  $\{t\} \times \partial\mathcal{D} \times \mathbb{R}^d$  and  $\Sigma_t^0$ .

**Lemma 3.5** (Carrillo [9], Lemma 2.3 and its proof).

(i) Let  $\psi \in \mathcal{Y}(Q_T)$ . Then  $\psi$  has trace values  $\gamma^+(\psi) \in L^2(\Sigma_T^+)$  (resp.  $\gamma^-(\psi) \in L^2(\Sigma_T^-)$ ) on  $\Sigma_T^+$  (resp. on  $\Sigma_T^-$ ). Moreover, for all  $t \in [0, T]$ ,  $\psi(t, \cdot)$  belongs to  $L^2(\mathcal{D} \times \mathbb{R}^d)$ .

(ii) For all  $\psi \in \mathcal{Y}(Q_t)$ , there exists a sequence  $\{\psi_n\}_{n \in \mathbb{N}}$  of  $\tilde{\mathcal{Y}}(Q_t)$  such that  $\psi_n$  tends to  $\psi$  in  $\|\cdot\|_{\mathcal{Y}(Q_t)}$  when  $n$  tends to  $+\infty$ . In particular,

$$\gamma^\pm(\psi) = \lim_{n \rightarrow +\infty} \psi_n \text{ in } L^2(\Sigma_t^\pm).$$

For the weighted spaces considered in this section, we reformulate the Green formula in Lemma 2.3 in [9], which is a direct consequence of the Lemma 3.5:

**Lemma 3.6.** *Let  $\psi \in \mathcal{H}(\omega, Q_T)$  be such that  $\mathcal{T}(\sqrt{\omega}\psi) \in \mathcal{H}'(Q_T)$ . Then  $\psi$  has traces  $\gamma^\pm(\psi) \in L^2(\omega, \Sigma_T^\pm)$ , and  $\psi(t, \cdot) \in L^2(\omega, \mathcal{D} \times \mathbb{R}^d)$ , such that the following holds for all  $\phi$  in  $\mathcal{Y}(Q_T)$ .*

$$\begin{aligned} & (\mathcal{T}(\psi), \phi)_{\mathcal{H}'(Q_t), \mathcal{H}(Q_t)} + (\mathcal{T}(\phi), \psi)_{\mathcal{H}'(Q_t), \mathcal{H}(Q_t)} \\ &= \int_{\mathcal{D} \times \mathbb{R}^d} \psi(t, x, u) \phi(t, x, u) dx du - \int_{\mathcal{D} \times \mathbb{R}^d} \psi(0, x, u) \phi(0, x, u) dx du \\ &+ \int_{\Sigma_t^+} (u \cdot n_{\mathcal{D}}(x)) \gamma^+(\psi)(s, x, u) \gamma^+(\phi)(s, x, u) d\lambda_\Sigma(s, x, u) \\ &+ \int_{\Sigma_t^-} (u \cdot n_{\mathcal{D}}(x)) \gamma^-(\psi)(s, x, u) \gamma^-(\phi)(s, x, u) d\lambda_\Sigma(s, x, u). \end{aligned} \quad (3.7)$$

For given  $q$ ,  $B$  and  $g$ , let us consider the linear Vlasov-Fokker-Planck equation:

$$\mathcal{T}(f) = \frac{\sigma^2}{2} \Delta_u f - (\nabla_u \cdot B f) + g, \quad \text{in } \mathcal{H}'(Q_T), \quad (3.8a)$$

$$f(0, x, u) = \rho_0(x, u), \quad \text{on } \mathcal{D} \times \mathbb{R}^d, \quad (3.8b)$$

$$\gamma^-(f)(t, x, u) = q(t, x, u), \quad \text{on } \Sigma_T^-. \quad (3.8c)$$

**Lemma 3.7.** *Assume  $(H_{\text{VFP}})$ -(ii). Then, given  $B \in L^\infty((0, T) \times \mathcal{D}; \mathbb{R}^d)$ ,  $q \in L^2(\omega, \Sigma_T^-)$ , and  $g \in L^2(\omega, Q_T)$ , there exists a unique solution  $f$  in  $V_1(\omega, Q_T)$  to (3.8). In addition, this solution admits trace functions  $\gamma^\pm(f)$  in  $L^2(\omega, \Sigma_T^\pm)$  and, for all  $t \in (0, T]$ ,*

$$\begin{aligned} & \|f(t)\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 + \sigma^2 \|\nabla_u f\|_{L^2(\omega, Q_t)}^2 + \|\gamma^+(f)\|_{L^2(\omega, \Sigma_t^+)}^2 \\ &= \|\rho_0\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 + \|q\|_{L^2(\omega, \Sigma_t^-)}^2 + \int_{Q_t} \left\{ \frac{\sigma^2}{2} \Delta_u \omega + (\nabla_u \omega \cdot B) \right\} |f|^2 + 2 \int_{Q_t} \omega g f. \end{aligned} \quad (3.9)$$

When  $g = 0$ , if  $\rho_0$  and  $q$  are nonnegative, then  $f$  and  $\gamma^+(f)$  are nonnegative.

*Proof.* In our situation of weighted spaces, it is easy to deduce from the original proof of Carrillo [9], that there exists a unique solution  $f \in \mathcal{H}(\omega, Q_T)$  to (3.8) and that  $\sqrt{\omega}f \in \mathcal{Y}(Q_T)$  (for the sake of completeness, the proof of this well-posedness result is given in the Appendix A.2).

Then, using Lemma 3.6, one can take  $\phi = \psi = \sqrt{\omega}f$  in the Green formula (3.7) and combined with (3.8a), we obtain that:

$$\begin{aligned} & (\mathcal{T}(\sqrt{\omega}f), \sqrt{\omega}f)_{\mathcal{H}'(Q_t), \mathcal{H}(Q_t)} = \|f(t)\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 - \|f(0)\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 + \|\gamma^+(f)\|_{L^2(\omega, \Sigma_t^+)}^2 - \|\gamma^-(f)\|_{L^2(\omega, \Sigma_t^-)}^2 \\ &= -\sigma^2 \|\nabla_u f\|_{L^2(\omega, Q_t)}^2 + \int_{Q_t} \left( \frac{\sigma^2}{2} \Delta_u \omega + (\nabla_u \omega \cdot B) \right) |f|^2 + 2 \int_{Q_t} \omega g f. \end{aligned}$$

Using (3.8b) and (3.8c), we deduce (3.9).

In order to conclude that  $f \in V_1(\omega, Q_T)$ , it remains to show the continuity of the mapping  $t \mapsto f(t)$  in  $L^2(\omega, \mathcal{D} \times \mathbb{R}^d)$ . Let us start by establishing right continuity. For  $0 < h < T$ , we define  $f_h(t, x, u) := f(t + h, x, u)$  on  $Q_{T-h}$ . Observe that, for  $\chi : [0, +\infty) \rightarrow [0, +\infty)$  in  $\mathcal{C}_b^1([0, +\infty))$ , one can check that, by (3.8a), (3.8c) and (3.8b),  $\chi(f_h - f) \in \mathcal{H}(\omega, Q_{T-h})$ ,  $\mathcal{T}(\chi(f_h - f)) \in \mathcal{H}'(Q_{T-h})$  and

$$\begin{aligned} & \mathcal{T}(\chi(f_h - f)) = \frac{\sigma^2}{2} \Delta_u \chi(f_h - f) - (\nabla_u \cdot B \chi(f_h - f)) + \chi(g_h - g) + \chi'(f_h - f), \quad \text{in } \mathcal{H}'(Q_{T-h}), \\ & \chi(f_h - f)(0, x, u) = \chi(0)(f(h, x, u) - f_0(x, u)), \quad \text{in } \mathcal{D} \times \mathbb{R}^d, \\ & \gamma^-(\chi(f_h - f))(t, x, u) = \chi(t)(q_h - q)(t, x, u), \quad \text{in } \Sigma_{T-h}^-. \end{aligned}$$

Since  $\chi(f_h - f) \in \mathcal{H}(\omega, Q_{T-h})$ , using (3.9), one obtains that, for all  $t \in (0, T)$

$$\begin{aligned} & \|\chi(t)(f_h - f)(t)\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 - \|\chi(0)(f(h) - f_0)\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 \\ & + \sigma^2 \|\nabla_u \chi(f_h - f)\|_{L^2(\omega, Q_{t+h})}^2 + \|\chi(\gamma^+(f_h) - \gamma^+(f))\|_{L^2(\omega, \Sigma_{t+h}^+)}^2 \\ & = \|\chi(g_h - q)\|_{L^2(\omega, \Sigma_{t-h}^-)}^2 + \int_{Q_{t+h}} \left( \frac{\sigma^2}{2} \Delta_u \omega + (\nabla_u \omega \cdot B) \right) |\chi(f_h - f)|^2 \\ & + \int_{Q_{t+h}} \chi' \chi \omega |f_h - f|^2 + 2 \int_{Q_{t+h}} \omega \chi(g_h - q)(f_h - f). \end{aligned} \quad (3.10)$$

Since  $\chi$  and  $\chi'$  are bounded, by using the estimations on  $\omega$  and its derivatives given in Lemma 3.4, and using Corollary A.1, all the terms above with an integral in time tend to 0 when  $h$  goes to 0. Hence we have, for a fixed  $t \in (0, T]$ ,

$$\lim_{h \rightarrow 0^+} \left| \|\chi(t)(f(t+h) - f(t))\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)} - \|\chi(0)(f(h) - f_0)\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)} \right| = 0.$$

We get the right continuity at time  $t$ , by taking  $\chi(0) = 0$  and  $\chi(t) = 1$ . The continuity at time  $t = 0$  is given by taking  $\chi(0) = 1$  and  $\chi(t) = 0$ . The left continuity is proved in an analogous way.  $\square$

### 3.2 The Maxwellian bounds for the linear Vlasov–Fokker–Planck equation

We state the existence of lower and upper-bounds for the solution to the linear problem

$$\begin{cases} \mathcal{T}(S) + (B \cdot \nabla_u S) - \frac{\sigma^2}{2} \Delta_u S = 0, & \text{in } \mathcal{H}'(Q_T), \\ S(0, x, u) = \rho_0(x, u), & \text{for } (x, u) \in \mathcal{D} \times \mathbb{R}^d, \\ S(t, x, u) = q(t, x, u), & \text{for } (t, x, u) \in \Sigma_T^- \end{cases} \quad (3.11)$$

in the following proposition.

**Proposition 3.8.** *Assume  $(H_{\text{VFP}})$ -(ii) and  $(H_{\text{VFP}})$ -(iii). For  $B \in L^\infty((0, T) \times \mathcal{D}; \mathbb{R}^d)$ , for  $(\underline{P}_0, \overline{P}_0)$  as in  $(H_{\text{VFP}})$ -(iii), let  $(\underline{p}, \overline{p})$  be a couple Maxwellian distributions with parameters  $(\underline{a}, \underline{\mu}, \underline{P}_0)$  and  $(\overline{a}, \overline{\mu}, \overline{P}_0)$  satisfying*

$$(a1) \quad \underline{\mu} > 1 \text{ and } \overline{\mu} \in (\tfrac{1}{2}, 1).$$

$$(a2) \quad \underline{a} \leq \frac{-\underline{\mu}}{2\sigma^2(\underline{\mu}-1)} \|B\|_{L^\infty((0,T) \times \mathcal{D}; \mathbb{R}^d)}^2 \text{ and } \overline{a} \geq \frac{\overline{\mu}}{2\sigma^2(1-\overline{\mu})} \|B\|_{L^\infty((0,T) \times \mathcal{D}; \mathbb{R}^d)}^2. \text{ Then the following properties hold:}$$

$$(d1) \quad \sup_{t \in [0, T]} \int_{\mathbb{R}^d} (1 + |u|) \omega(u) |\overline{p}(t, u)|^2 du < +\infty, \quad \text{and} \quad \inf_{t \in [0, T]} \int_{\mathbb{R}^d} \underline{p}(t, u) du > 0.$$

$$(d2) \quad \text{Let } S \text{ be the unique weak solution of (3.11) with data } \rho_0, q \text{ and } B. \text{ If } \underline{p} \leq q \leq \overline{p}, \lambda_\Sigma\text{-a.e. on } \Sigma_T^-, \text{ then } \underline{p} \leq S \leq \overline{p}, \text{ a.e. on } Q_T, \text{ and } \underline{p} \leq \gamma^+(S) \leq \overline{p}, \lambda_\Sigma\text{-a.e. on } \Sigma_T^+.$$

For the proof of Proposition 3.8, we will use the notions of super-solution and sub-solution of Maxwellian type related to the operator

$$\mathcal{L}_B(\psi) = \mathcal{T}(\psi) + (B \cdot \nabla_u \psi) - \frac{\sigma^2}{2} \Delta_u \psi. \quad (3.12)$$

**Definition 3.9.** *Let  $P$  be a Maxwellian distribution with parameters  $(a, \mu, P_0)$ . For  $B \in L^\infty((0, T) \times \mathcal{D}; \mathbb{R}^d)$ , we say that*

1.  *$P$  is a super-solution of Maxwellian type for  $\mathcal{L}_B$  if  $0 \leq \mathcal{L}_B(P) < +\infty$ , a.e. on  $Q_T$ .*

2.  *$P$  is a sub-solution of Maxwellian type for  $\mathcal{L}_B$  if  $-\infty < \mathcal{L}_B(P) \leq 0$ , a.e. on  $Q_T$ .*

The proof of Proposition 3.8 proceeds as follows. First, we exhibit a class of Maxwellian distributions satisfying the properties (d1) and some regularity properties (see Lemma 3.10). Second, we establish a comparison principle between super-solutions and sub-solutions of Maxwellian type and the weak solution to (3.11) (see Lemma 3.12). We thus deduce a particular class of Maxwellian distributions satisfying the properties ((d1), (d2)).

Finally, we identify the class of super-solutions and sub-solutions of Maxwellian type for the operator  $\mathcal{L}_B$  for all  $B$  fixed in  $L^\infty((0, T) \times \mathcal{D}; \mathbb{R}^d)$  (see Lemma 3.12). Combining these results, we conclude on Proposition 3.8.

**Step 1.** We start by emphasizing some technical properties of the Maxwellian distributions.

**Lemma 3.10.** *Let  $p$  be a Maxwellian distribution equipped with the parameters  $(a, \mu, p_0)$  such that  $2\mu > 1$ ,  $p_0$  is not identically equal to zero, and*

$$\int_{\mathbb{R}^d} (1 + |u|) \omega(u) |p_0(u)|^2 du < +\infty. \quad (3.13)$$

*Then the following properties hold:*

- (i1)  $\sup_{t \in [0, T]} \int_{\mathbb{R}^d} (1 + |u|) \omega(u) |p(t, u)|^2 du < +\infty$ .
- (i2)  $\inf_{t \in [0, T]} \int_{\mathbb{R}^d} p(t, u) du > 0$ , if  $\mu > 1$ .
- (i3) *There exists a sequence of positive reals  $\{\epsilon_k ; k \in \mathbb{N}\}$  such that*

$$\lim_{k \rightarrow +\infty} \epsilon_k = 0 \text{ and } \lim_{k \rightarrow +\infty} p(\epsilon_k, \cdot) = p_0(\cdot) \text{ in } L^2(\mathbb{R}^d).$$

- (i4) *For all  $\delta > 0$ ,  $\partial_t p$  belongs to  $L^2((\delta, T) \times \mathbb{R}^d)$ .*
- (i5)  $p \in \mathcal{H}(Q_T)$ .

The proof of this lemma relies on some well-known properties of Gaussian distributions and is postponed in the Appendix section. Lemma 3.10 enables us to identify the class of Maxwellian distributions satisfying the properties (d1) in Proposition 3.8. The properties ((i3), (i4), (i5)) emphasize regularities that we will need in the sequel.

**Step 2: Comparison principle and Maxwellian bounds.**

**Lemma 3.11.** *Let  $B \in L^\infty((0, T) \times \mathcal{D}; \mathbb{R}^d)$  be fixed and let  $\underline{p}$ ,  $\bar{p}$  be two Maxwellian distributions, sub-solution and super-solution for  $\mathcal{L}_B$  respectively with parameters  $(\underline{a}, \underline{\mu}, \underline{p}_0)$  and  $(\bar{a}, \bar{\mu}, \bar{p}_0)$  such that  $2\underline{\mu} \wedge 2\bar{\mu} > 1$  and  $\underline{p}_0$ ,  $\bar{p}_0$  satisfying (3.13). If  $\underline{p}_0 \leq \rho_0 \leq \bar{p}_0$ , a.e. on  $\mathcal{D} \times \mathbb{R}^d$ , and  $\underline{p} \leq q \leq \bar{p}$ ,  $\lambda_\Sigma$ -a.e. on  $\Sigma_T^-$ , then we have*

$$\begin{aligned} \underline{p} &\leq S \leq \bar{p}, \text{ a.e. on } Q_T, \\ \underline{p} &\leq \gamma^+(S) \leq \bar{p}, \text{ } \lambda_\Sigma\text{-a.e. on } \Sigma_T^+, \end{aligned} \quad (3.14)$$

for  $S$  the weak solution in  $V_1(\omega, Q_T)$  to (3.11) with data  $(\rho_0, q, B)$ .

*Proof.* Let us first prove the implication

$$\begin{cases} \rho_0 \leq \bar{p}_0, \text{ a.e. on } (0, T) \times \mathcal{D}, \\ q \leq \bar{p}, \text{ } \lambda_\Sigma\text{-a.e. on } \Sigma_T^-. \end{cases} \implies \begin{cases} S \leq \bar{p}, \text{ a.e. on } Q_T, \\ \gamma^+(S) \leq \bar{p}, \text{ } \lambda_\Sigma\text{-a.e. on } \Sigma_T^+. \end{cases} \quad (3.15)$$

Defining  $F$ ,  $\gamma^+(F)$  with

$$\begin{aligned} F(t, x, u) &= (\bar{p}(t, u) - S(t, x, u)), \\ \gamma^+(F)(t, x, u) &= (\bar{p}(t, u) - \gamma^+(S)(t, x, u)), \end{aligned}$$

(3.15) is equivalent to the inequality:  $\forall t \in (0, T]$ ,

$$\| (F(t))^- \|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \| (\gamma^+(F))^- \|_{L^2(\Sigma_t^+)}^2 \leq \| (\bar{p}_0 - \rho_0)^- \|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \| (\bar{p} - q)^- \|_{L^2(\Sigma_t^-)}^2. \quad (3.16)$$

In order to obtain (3.16), we establish a Green's identity on a smooth approximation of  $(F)^-$ . For fixed  $t$  in  $(0, T]$ , by Lemma 3.5, there exists a sequence of  $\mathcal{C}_c^\infty(\overline{Q_t})$ -functions,  $\{f_n\}_{n \in \mathbb{N}}$ , such that

$$\begin{aligned} \lim_{n \rightarrow +\infty} f_n &= S \text{ on } \mathcal{H}(Q_t), \quad \lim_{n \rightarrow +\infty} \mathcal{T}(f_n) = \mathcal{T}(S) \text{ on } \mathcal{H}'(Q_t), \\ \lim_{n \rightarrow +\infty} \int_{\mathcal{D} \times \mathbb{R}^d} |f_n(s, x, u) - S(s, x, u)|^2 dx du &= 0, \quad \forall s \in [0, t], \\ \lim_{n \rightarrow +\infty} \int_{\Sigma_t^\pm} |(u \cdot n_{\mathcal{D}}(x))| |f_n(s, x, u) - \gamma^\pm(S)(s, x, u)|^2 d\lambda_\Sigma(s, x, u) &= 0. \end{aligned} \quad (3.17)$$

In addition, according to Lemma 3.10,  $\bar{p}$  satisfy the properties (i1) to (i5). Let us define the sequence of  $\mathcal{C}_b^2(Q_t)$ -functions  $\{F_n\}_{n \in \mathbb{N}}$  by

$$F_n(s, x, u) = \bar{p}(s, x, u) - f_n(s, x, u).$$

Then by definition of  $\mathcal{L}_B$ , for a.e.  $(s, x, u)$  in  $Q_t$ ,

$$\begin{aligned} \mathcal{T}(F_n)(s, x, u) &= \mathcal{L}_B(\bar{p} - f_n)(s, x, u) - (B(s, x) \cdot \nabla_u F_n(s, x, u)) + \frac{\sigma^2}{2} \Delta_u F_n(s, x, u) \\ &\geq -\mathcal{L}_B(f_n)(s, x, u) - (B(s, x) \cdot \nabla_u F_n(s, x, u)) + \frac{\sigma^2}{2} \Delta_u F_n(s, x, u) \end{aligned} \quad (3.18)$$

since  $\mathcal{L}_B(\bar{p} - f_n) \geq -\mathcal{L}_B(f_n)$  as  $\bar{p}$  is a super-solution for  $\mathcal{L}_B$ . Using the sequence  $\{\epsilon_k; k \in \mathbb{N}\}$  given by (i3) and by taking  $k$  such that  $0 < \epsilon_k \leq t$ , (i4) and (i5) ensure that  $\mathcal{T}(F_n) \in L^2(Q_{\epsilon_k, t})$  and  $F_n \in \mathcal{H}(Q_{\epsilon_k, t})$  for  $Q_{\epsilon_k, t} := (\epsilon_k, t) \times \mathcal{D} \times \mathbb{R}^d$ . These properties are also true for  $(F_n)^-$  (see Theorem A.2). Multiplying both sides of (3.18) by  $(F_n)^-$ , and integrating the resulting expression on  $Q_{\epsilon_k, t}$ , we obtain

$$\int_{Q_{\epsilon_k, t}} \mathcal{T}(F_n)(F_n)^- \geq - \int_{Q_{\epsilon_k, t}} (B \cdot \nabla_u F_n)(F_n)^- + \frac{\sigma^2}{2} \int_{Q_{\epsilon_k, t}} (\Delta_u F_n)(F_n)^- - \int_{Q_{\epsilon_k, t}} \mathcal{L}_B(f_n)(F_n)^-. \quad (3.19)$$

For the l.h.s in (3.19), an integration by parts yields,

$$\begin{aligned} \int_{Q_{\epsilon_k, t}} \mathcal{T}(F_n)(F_n)^- &= - \int_{Q_{\epsilon_k, t}} \left\{ \partial_t (F_n)^- + (u \cdot \nabla_x (F_n)^-) \right\} (F_n)^- \\ &= \frac{1}{2} \| (F_n(\epsilon_k))^- \|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 - \frac{1}{2} \| (F_n(t))^- \|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 - \frac{1}{2} \| (F_n)^- \|_{L^2(\Sigma_{\epsilon_k, t}^+)}^2 + \frac{1}{2} \| (F_n)^- \|_{L^2(\Sigma_{\epsilon_k, t}^-)}, \end{aligned}$$

with  $\Sigma_{\epsilon_k, t}^\pm := \{(s, x, u) \in \Sigma_t^\pm; s \in (\epsilon_k, t)\}$ . In the r.h.s. in (3.19), as  $B$  depends only on  $x$ , we get

$$- \int_{Q_{\epsilon_k, t}} (B \cdot \nabla_u F_n)(F_n)^- + \frac{\sigma^2}{2} \int_{Q_{\epsilon_k, t}} (\Delta_u F_n)(F_n)^- = \frac{\sigma^2}{2} \int_{Q_{\epsilon_k, t}} |\nabla_u (F_n)^-|^2 \geq 0.$$

Coming back to (3.19), it follows that

$$\begin{aligned} &\| (F_n(\epsilon_k))^- \|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \| (F_n)^- \|_{L^2(\Sigma_{\epsilon_k, t}^-)}^2 \\ &\geq \| (F_n(t))^- \|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \| (F_n)^- \|_{L^2(\Sigma_{\epsilon_k, t}^+)}^2 - 2 \int_{Q_{\epsilon_k, t}} \mathcal{L}_B(f_n)(F_n)^-. \end{aligned}$$

Taking the limit  $k \rightarrow +\infty$ , (i3) implies that  $\lim_{k \rightarrow +\infty} |(F_n(\epsilon_k))^-| = |(\bar{p}_0 - f_n(0))^-|$ . Thus

$$\begin{aligned} &\| (\bar{p}_0 - f_n(0))^- \|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \| (F_n)^- \|_{L^2(\Sigma_t^-)}^2 \\ &\geq \| (F_n(t))^- \|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \| (F_n)^- \|_{L^2(\Sigma_t^+)}^2 - 2 \int_{Q_t} \mathcal{L}_B(f_n)(F_n)^- \end{aligned} \quad (3.20)$$

It remain to study the limit w.r.t.  $n$ . By (3.17),

$$\lim_{n \rightarrow +\infty} \left( \| (\bar{p}_0 - f_n(0))^- \|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \| (F_n)^- \|_{L^2(\Sigma_t^-)}^2 \right) = \| (\bar{p}_0 - \rho_0)^- \|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \| (p - g)^- \|_{L^2(\Sigma_t^-)}^2,$$

$$\lim_{n \rightarrow +\infty} \left( \| (F_n(t))^- \|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \| (F_n)^- \|_{L^2(\Sigma_t^+)}^2 \right) = \| (F(t))^- \|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \| (\gamma^+(F))^- \|_{L^2(\Sigma_t^+)}^2.$$

For the last term in (3.20), an integration by parts yields

$$\int_{Q_t} \mathcal{L}_B(f_n) (F_n)^- = \left( \mathcal{T}(f_n), (\bar{p} - f_n)^- \right)_{\mathcal{H}'(Q_t), \mathcal{H}(Q_t)} + \int_{Q_t} (B \cdot \nabla_u f_n) (\bar{p} - f_n)^- + \frac{\sigma^2}{2} \int_{Q_t} (\nabla_u f_n \cdot \nabla_u (\bar{p} - f_n)^-) = 0.$$

As  $\lim_{n \rightarrow +\infty} (\bar{p} - f_n)^- = (\bar{p} - S)^-$  in  $\mathcal{H}(Q_T)$  and  $\lim_{n \rightarrow +\infty} \mathcal{T}(f_n) = \mathcal{T}(S)$  in  $\mathcal{H}'(Q_T)$ , we get

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_{Q_t} \mathcal{L}_B(f_n) (F_n)^- \\ &= \left( \mathcal{T}(S), (\bar{p} - S)^- \right)_{\mathcal{H}'(Q_t), \mathcal{H}(Q_t)} + \int_{Q_t} (B \cdot \nabla_u S) (\bar{p} - S)^- + \frac{\sigma^2}{2} \int_{Q_t} (\nabla_u S \cdot \nabla_u (\bar{p} - S)^-) = 0 \end{aligned}$$

since  $\mathcal{T}(S) + (B \cdot \nabla_u S) - \frac{\sigma^2}{2} \Delta_u S = 0$  in  $\mathcal{H}'(Q_T)$ . Coming back to (3.20), we get

$$\| (F(t))^- \|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \| (F)^- \|_{L^2(\Sigma_t^+)}^2 \leq \| (\bar{p}_0 - \rho_0)^- \|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \| (\bar{p} - q)^- \|_{L^2(\Sigma_t^-)}^2$$

which gives (3.16). The symmetric implication

$$\left\{ \begin{array}{l} \underline{p}_0 \leq \rho_0, \text{ a.e. on } (0, T) \times \mathcal{D}, \\ \underline{p} \leq q, \text{ } \lambda_{\Sigma}\text{-a.e. on } \Sigma_T^-. \end{array} \right\} \implies \left\{ \begin{array}{l} \underline{p} \leq S, \text{ a.e. on } Q_T, \\ \underline{p} \leq \gamma^+(S), \text{ } \lambda_{\Sigma}\text{-a.e. on } \Sigma_T^+. \end{array} \right. \quad (3.21)$$

is proved in the same way: define

$$\begin{aligned} J(t, x, u) &:= (S(t, x, u) - \underline{p}(t, u)), \\ \gamma^+(J)(t, x, u) &:= (\gamma^+(S)(t, x, u) - \underline{p}(t, u)), \end{aligned}$$

we may establish that

$$\| (\rho_0 - \underline{p}_0)^- \|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \| (q - \underline{p})^- \|_{L^2(\Sigma_T^-)}^2 \geq \| (J(t))^- \|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \| (\gamma^+(J))^- \|_{L^2(\Sigma_T^+)}^2, \quad (3.22)$$

and we conclude (3.21). The inequality (3.22) is proved by using the sequence

$$J_n(s, x, u) := f_n(s, x, u) - \underline{p}(s, u),$$

with  $\{f_n\}_{n \in \mathbb{N}}$  satisfying (3.17), and using the fact that  $\underline{p}$  is a sub-solution of Maxwellian type, we obtain: for all  $t \in (0, T]$  fixed, for a.e.  $(s, x, u)$  in  $Q_t$ , it holds

$$\mathcal{T}(J_n)(s, x, u) \geq \mathcal{L}_B(f_n)(s, x, u) - (B(s, x) \cdot \nabla_u J_n(s, x, u)) + \frac{\sigma^2}{2} \Delta_u J_n(s, x, u).$$

Replicating the arguments in (3.15), we get

$$\| (f_n(0) - \underline{p}_0)^- \|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \| (J_n)^- \|_{L^2(\Sigma_t^-)}^2 \geq \| (J_n(t))^- \|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 + \| (J_n)^- \|_{L^2(\Sigma_t^+)}^2 + 2 \int_{Q_{\epsilon_k, t}} \mathcal{L}_B(f_n) (J_n)^-$$

We obtain (3.22) by taking the limit  $n \rightarrow +\infty$ . □

### Step 3: Existence of sub- and super-solutions of Maxwellian type.

**Lemma 3.12.** *Let  $p$  be a Maxwellian distribution with parameters  $(a, \mu, p_0)$ . For  $B \in L^\infty((0, T) \times \mathcal{D}; \mathbb{R}^d)$ , let  $\mathcal{L}_B$  be the operator defined as in (3.12). Then the following properties hold.*

- (i) *If  $\mu \in (0, 1)$  and  $a \geq \frac{\mu}{2\sigma^2(1-\mu)} \|B\|_{L^\infty((0, T) \times \mathcal{D}; \mathbb{R}^d)}^2$  then  $p$  is a super-solution of Maxwellian type for  $\mathcal{L}_B$ .*

(ii) If  $\mu > 1$  and  $a \leq \frac{-\mu}{2\sigma^2(\mu-1)} \|B\|_{L^\infty((0,T) \times \mathcal{D}; \mathbb{R}^d)}^2$  the  $p$  is a sub-solution of Maxwellian type for  $\mathcal{L}_B$ .

*Proof.* By the special form (3.3) of the considered Maxwellian distribution, we have

$$\begin{aligned} \mathcal{L}_B(p)(t, x, u) &= a \exp\{at\} m^\mu(t, u) + \mu \exp\{at\} \left( \partial_t m(t, u) - \frac{\sigma^2}{2} \Delta_u m(t, u) \right) m^{\mu-1}(t, u) \\ &\quad + \mu \exp\{at\} (B(t, x) \cdot \nabla_u m(t, u)) m^{\mu-1}(t, u) - \frac{\sigma^2}{2} \mu(\mu-1) \exp\{at\} |\nabla_u m(t, u)|^2 m^{\mu-2}(t, u). \end{aligned}$$

Since  $m$  is a classical solution of the heat equation, the preceding equality reduces to

$$\mathcal{L}_B(p)(t, x, u) = \exp\{at\} m^{\mu-2}(t, u) \left[ a |m(t, u)|^2 - \frac{\sigma^2}{2} \mu(\mu-1) |\nabla_u m(t, u)|^2 + \mu (B(t, x) \cdot \nabla_u m(t, u)) m(t, u) \right].$$

The sign of  $\mathcal{L}_B(p)$  is thus determined by the function:

$$J(t, x, u) := a |m(t, u)|^2 - \frac{\sigma^2 \mu(\mu-1)}{2} |\nabla_u m(t, u)|^2 + \mu (B(t, x) \cdot \nabla_u m(t, u)) m(t, u). \quad (3.23)$$

• When  $a$  and  $\mu$  satisfy (i), using the identity  $(u_1 \cdot u_2) = \frac{1}{2} \left| \epsilon u_1 + \frac{u_2}{\epsilon} \right|^2 - \frac{\epsilon^2 |u_1|^2}{2} - \frac{|u_2|^2}{2\epsilon^2}$ , for  $u_1, u_2 \in \mathbb{R}^d$ ,  $\epsilon > 0$ , we have

$$\mu (B(t, x) \cdot \nabla_u m(t, u)) m(t, u) = \frac{1}{2} \left| \epsilon \nabla_u m(t, u) + \frac{1}{\epsilon} m(t, u) B(t, x) \right|^2 - \frac{1}{\epsilon^2} |m(t, u) B(t, x)|^2 - \epsilon^2 |\nabla_u m(t, u)|^2.$$

Inserting this equality into (3.23), with  $\epsilon = \frac{\sigma \sqrt{1-\mu}}{\sqrt{\mu}}$  ( $> 0$  since  $0 < \mu < 1$ ), it follows that

$$J(t, x, u) = \left( a - \frac{\mu}{2\sigma^2(1-\mu)} |B(t, x)|^2 \right) |m(t, u)|^2 + \frac{1}{2} \left| \sigma \sqrt{\mu(1-\mu)} \nabla_u m(t, u) + \frac{\sqrt{\mu}}{\sigma \sqrt{1-\mu}} m(t, u) B(t, x) \right|^2,$$

where, since (i) is satisfied,

$$a - \frac{\mu}{2\sigma^2(1-\mu)} |B(t, x)|^2 \geq a - \frac{\mu}{2\sigma^2(1-\mu)} \|B\|_{L^\infty((0,T) \times \mathbb{R}^d; \mathbb{R}^d)}^2 \geq 0.$$

We thus deduce that  $J$  (and consequently  $\mathcal{L}_B(p)$ ) is nonnegative in the situation (i).

• When  $a$  and  $\mu$  satisfy the assumption in (ii), using the identity  $(u_1 \cdot u_2) = -\frac{1}{2} \left| \epsilon u_1 - \frac{u_2}{\epsilon} \right|^2 + \frac{\epsilon^2 |u_1|^2}{2} + \frac{|u_2|^2}{2\epsilon^2}$ , for  $u_1, u_2 \in \mathbb{R}^d$ ,  $\epsilon > 0$ , we get

$$\mu (B(t, x) \cdot \nabla_u m(t, u)) m(t, u) = \frac{1}{2\epsilon^2} |B(t, x)|^2 |m(t, u)|^2 + \frac{\epsilon^2}{2} \mu^2 |\nabla_u m(t, u)|^2 - \frac{1}{2} \left| \epsilon m(t, u) B(t, x) - \frac{\mu}{\epsilon} \nabla_u m(t, u) \right|^2.$$

Taking  $\epsilon = \frac{\sigma \sqrt{\mu-1}}{\sqrt{\mu}}$  it follows that

$$J(t, x, u) = \left( a + \frac{\mu}{2\sigma^2(\mu-1)} |B(t, x)|^2 \right) |m(t, u)|^2 - \frac{1}{2} \left| \frac{\sigma \sqrt{\mu-1}}{\sqrt{\mu}} m(t, u) B(t, x) - \frac{\mu \sqrt{\mu}}{\sigma \sqrt{\mu-1}} \nabla_u m(t, u) \right|^2.$$

Since (ii) ensures that

$$a + \frac{\mu}{2\sigma^2(1-\mu)} |B(t, x)|^2 \leq a + \frac{\mu}{2\sigma^2(1-\mu)} \|B\|_{L^\infty((0,T) \times \mathcal{D}; \mathbb{R}^d)}^2 \leq 0,$$

we conclude that  $\mathcal{L}_B(p)$  is non-positive.  $\square$

**Step 4: Proof of Proposition 3.8.** Let  $(\underline{p}, \overline{p})$  be a couple of Maxwellian distributions such that  $(\underline{p}_0, \overline{p}_0)$  are given as in  $(H_{\text{VFP}})$ -(iii) and such that their parameters  $(\underline{a}, \underline{\mu})$  and  $(\overline{a}, \overline{\mu})$  satisfy the properties (a1) and (a2) in Proposition 3.8. Applying Lemma 3.12,  $(\underline{p}, \overline{p})$  define respectively a sub-solution and a super-solution of Maxwellian type for the linear operator. Moreover, recalling that  $\underline{p}_0, \overline{p}_0 \in L^2(\omega, Q_T)$  are positives, Lemma 3.7 implies that these Maxwellian distributions satisfies the conditions of Lemma 3.10 and Lemma 3.11. We thus deduce that  $(\underline{p}, \overline{p})$  verify the properties (d1) to (d2) of Proposition 3.8.

### 3.3 Construction of a weak solution to the conditional McKean-Vlasov-Fokker-Planck equation

**Step 1: Introduction of the specular boundary condition.** We consider now a linear problem endowing a given convection term  $B \in L^\infty((0, T) \times \mathcal{D}; \mathbb{R}^d)$  and submitted to the specular boundary condition:

$$\begin{cases} \mathcal{T}(S) + (B \cdot \nabla_u S) - \frac{\sigma^2}{2} \Delta_u S = 0, & \text{in } \mathcal{H}'(Q_T), \\ S(0, x, u) = \rho_0(x, u), & \text{on } \mathcal{D} \times \mathbb{R}^d, \\ S(t, x, u) = S(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), & \text{on } \Sigma_T^-. \end{cases} \quad (3.24)$$

**Proposition 3.13.** Assume  $(H_{\text{VFP}})$ -(ii) and  $(H_{\text{VFP}})$ -(iii), and assume that  $B \in L^\infty((0, T) \times \mathcal{D}; \mathbb{R}^d)$  is given. Let  $(\underline{P}, \overline{P})$  couple Maxwellian distributions with parameters  $(\underline{a}, \underline{\mu}, \underline{P}_0)$  and  $(\overline{a}, \overline{\mu}, \overline{P}_0)$  respectively, verifying the hypotheses of Proposition 3.8. Then there exists a unique weak solution  $\mathcal{S}$  in  $V_1(\omega, Q_T)$  of (3.24) such that

$$\begin{aligned} \underline{P}(t, u) &\leq \mathcal{S}(t, x, u) \leq \overline{P}(t, u), \text{ for a.e. } (t, x, u) \in Q_T, \\ \underline{P}(t, u) &\leq \gamma^\pm(\mathcal{S})(t, x, u) \leq \overline{P}(t, u), \text{ for } \lambda_\Sigma\text{-a.e. } (t, x, u) \in \Sigma_T^\pm. \end{aligned}$$

*Proof.* For the existence claim, let us introduce the functional space

$$E = \{ \psi \in V_1(\omega, Q_T) ; \psi \text{ admits trace functions } \gamma^\pm(\psi) \text{ on } \Sigma_T^\pm \text{ belonging to } L^2(\omega, \Sigma_T) \},$$

equipped with the norm

$$\|\psi\|_E^2 = \|\psi\|_{V_1(\omega, Q_T)}^2 + \|\gamma^+(\psi)\|_{L^2(\omega, \Sigma_T^+)}^2 + \|\gamma^-(\psi)\|_{L^2(\omega, \Sigma_T^-)}^2.$$

For all  $f \in E$ , we denote by  $S(f)$  the unique weak solution (in the sense of Lemma 3.7) to the linear Vlasov-Fokker-Planck

$$\begin{cases} \mathcal{T}(S(f)) + (B \cdot \nabla_u S(f)) - \frac{\sigma^2}{2} \Delta_u S(f) = 0, & \text{in } \mathcal{H}'(Q_T), \\ S(f)(0, x, u) = \rho_0(x, u), & \text{for } (x, u) \in \mathcal{D} \times \mathbb{R}^d, \\ S(f)(t, x, u) = \gamma^+(f)(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)) & \text{on } \Sigma_T^-. \end{cases} \quad (3.25)$$

Lemma 3.7 ensures that  $S(f) \in V_1(\omega, Q_T)$ , and that the trace functions  $\gamma^\pm(S(f))$  belong to  $L^2(\omega, \Sigma_T^\pm)$ . Therefore, we can define the mapping

$$S : f \in E \longrightarrow S(f) \in E.$$

If  $S$  admits a fixed point  $\mathcal{S}$ , then it naturally satisfies the specular boundary condition

$$\gamma^-(\mathcal{S})(t, x, u) = \gamma^+(\mathcal{S})(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), \text{ for } (t, x, u) \in \Sigma_T^-,$$

implying that  $\mathcal{S}$  is a weak solution to (3.24). In order to establish the existence of this fixed point, let us observe the following properties of  $S$  (the proof of these properties will be described later):

- (1)  $S$  is Lipschitz-continuous on  $E$ ;

(2)  $S$  is a increasing mapping on  $E$  w.r.t. the following order relation:

$$f_1 \leq f_2 \text{ on } E \iff \begin{cases} f_1(t, x, u) \leq f_2(t, x, u), \text{ for a.e. } (t, x, u) \in Q_T, \\ \gamma(f_1)(t, x, u) \leq \gamma(f_2)(t, x, u), \text{ } \lambda_\Sigma\text{-for a.e. } (t, x, u) \in \Sigma_T; \end{cases}$$

(3) If  $\underline{P} \leq \gamma^+(f) \leq \overline{P}$ ,  $\lambda_\Sigma$ -a.e. on  $\Sigma_T^+$ , then  $\underline{P} \leq S(f) \leq \overline{P}$  on  $E$ .

The fixed point of  $\mathcal{S}$  will now arise from the convergence of the sequence  $\{S_n\}_{n \in \mathbb{N}}$  defined iteratively by  $\{S_0 = \underline{P}, S_{n+1} = S(S_n)\}$ .

Indeed, the monotone property (2) of  $S$  implies that  $\{S_n\}_{n \in \mathbb{N}}$  is increasing in  $E$ . In addition, since  $\underline{P} = S_0 \leq \overline{P}$ , Proposition 3.8 ensures that  $\underline{P} \leq S_1 \leq \overline{P}$  and  $\underline{P} \leq \gamma^+(S_1) \leq \overline{P}$ . Using repeatedly the property (3), it holds that  $\forall n \in \mathbb{N}$ ,

$$\underline{P} \leq S_n \leq \overline{P}, \text{ on } E. \quad (3.26)$$

Let us assume that  $\underline{P} \leq S_n \leq \overline{P}$  on  $E$  for  $n \geq 0$ . (3) ensure that the (Maxwellian) bounds on  $S_n$  propagate to  $S_{n+1}$  on  $E$ .

$\{S_n\}_{n \in \mathbb{N}}$  being increasing and uniformly bounded on  $E$ , we deduce that the sequence  $\{S_n\}_{n \in \mathbb{N}}$  and  $\{\gamma(S_n)\}_{n \in \mathbb{N}}$  converge. Thus, set

$$\mathcal{S}(t, x, u) := \lim_{n \rightarrow +\infty} S_n(t, x, u), \text{ for } (t, x, u) \in Q_T,$$

and

$$\gamma^\pm(\mathcal{S})(t, x, u) = \lim_{n \rightarrow +\infty} \gamma^\pm(S_n)(t, x, u), \text{ for } (t, x, u) \in \Sigma_T^\pm.$$

According to (3.26),  $\underline{P} \leq \mathcal{S} \leq \overline{P}$  on  $E$ . Since  $\overline{P} \in L^2(\omega, Q_T)$  and  $\overline{P} \in L^2(\omega, \Sigma_T)$ , by dominated convergence, we obtain that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \|\mathcal{S} - S_n\|_{L^2(\omega, Q_T)} &= 0, \\ \lim_{n \rightarrow +\infty} \|\gamma^\pm(\mathcal{S}) - \gamma^\pm(S_n)\|_{L^2(\omega, \Sigma_T^\pm)} &= 0. \end{aligned}$$

Owing to the continuity of  $S$  given in (1), we deduce that

$$\begin{aligned} \mathcal{S} &= \lim_{n \rightarrow +\infty} S_{n+1} = \lim_{n \rightarrow +\infty} S(S_n) = S(\mathcal{S}), \\ \gamma^\pm(\mathcal{S}) &= \lim_{n \rightarrow +\infty} \gamma^\pm(S_{n+1}) = \lim_{n \rightarrow +\infty} \gamma^\pm(S(S_n)) = \gamma^\pm(S(\mathcal{S})). \end{aligned}$$

We furthermore observe that, owing to (3.26), (3.9) ensures that

$$\sup_n \|\nabla_u S_n\|_{L^2(\omega, Q_t)}^2 \leq C \left( \|\rho_0\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 + \max_{t \in (0, T)} \|\overline{P}(t)\|_{L^2(\omega, \mathbb{R}^d)} \right)$$

where  $C$  is some constant depending only on  $d, T, \sigma, \|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}, \alpha$  and the Lebesgue measure of  $\mathcal{D}$ . It follows that  $\mathcal{S} \in \mathcal{H}(\omega, Q_T)$  with  $\nabla_u \mathcal{S} = \lim_{n \rightarrow +\infty} \nabla_u S_n$  in  $L^2(\omega, Q_T)$ . We thus conclude that  $\mathcal{S}$  has a fixed point  $S$  in  $E$ . For the uniqueness of weak solution to (3.24), consider two weak solutions  $\mathcal{S}^1, \mathcal{S}^2$ , to Equation (3.24). Set

$$\begin{aligned} R(t, x, u) &:= \mathcal{S}^1(t, x, u) - \mathcal{S}^2(t, x, u) \text{ for } (t, x, u) \in Q_T, \\ \gamma^\pm(R)(t, x, u) &:= (\gamma^\pm(\mathcal{S}^1) - \gamma^\pm(\mathcal{S}^2))(t, x, u) \text{ for } (t, x, u) \in \Sigma_T^\pm. \end{aligned}$$

By definition,  $R$  and  $\gamma^\pm(R)$  satisfy the equation:

$$\begin{aligned} \mathcal{T}(R) + (B \cdot \nabla_u R) - \frac{\sigma^2}{2} \Delta_u R &= 0 \text{ on } Q_T, \\ R(0, x, u) &= \mathcal{S}^1(0) - \mathcal{S}^2(0) = 0 \text{ for } (x, u) \in \mathcal{D} \times \mathbb{R}^d, \\ \gamma^+(R)(t, x, u) &= (\gamma^+(\mathcal{S}^1) - \gamma^+(\mathcal{S}^2))(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)) \text{ on } \Sigma_T^-. \end{aligned}$$

Using Lemma 3.7, one has: for all  $t \in (0, T]$ ,

$$\begin{aligned} & \|R(t)\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 + \sigma^2 \|\nabla_u R\|_{L^2(\omega, Q_T)}^2 + \|\gamma^+(R)\|_{L^2(\omega, \Sigma_t^+)}^2 \\ &= \|\gamma^-(R)\|_{L^2(\omega, \Sigma_t^-)}^2 - \int_{Q_t} \left( \frac{\sigma^2}{2} \Delta_u \omega + (B \cdot \nabla_u \omega) \right) |R|^2. \end{aligned}$$

$R$  and  $\gamma(R)$  still verify the inequality (3.27). Since  $\mathcal{S}^1$  and  $\mathcal{S}^2$  verify the specular boundary condition,

$$\begin{aligned} & \int_{\Sigma_t^-} (u \cdot n_{\mathcal{D}}(x)) \omega(u) |\gamma^+(R)(s, x, u)|^2 d\lambda_{\Sigma}(s, x, u) \\ &= - \int_{\Sigma_t^+} (u \cdot n_{\mathcal{D}}(x)) \omega(u) |\gamma^+(R)(s, x, u)|^2 d\lambda_{\Sigma}(s, x, u), \end{aligned}$$

so that, using Lemma 3.4, the preceding inequality is reduced to

$$\|R\|_{V_1(\omega, Q_t)}^2 \leq \left( \alpha d + 2\alpha \left( \frac{\alpha}{2} - 1 \right) + \alpha \|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)} \right) \int_0^t \|R\|_{V_1(\omega, Q_s)}^2 ds.$$

This ensures the uniqueness of solutions by applying Gronwall's Lemma.

**Proof of (1).** For  $f_1, f_2 \in E$ , let us set

$$\begin{aligned} R(t, x, u) &= S(f_1)(t, x, u) - S(f_2)(t, x, u), \text{ for } (t, x, u) \in Q_T, \\ \gamma^\pm(R)(t, x, u) &= (\gamma^\pm(S(f_1)) - \gamma^\pm(S(f_2)))(t, x, u), \text{ for } (t, x, u) \in \Sigma_T^\pm. \end{aligned}$$

By definition,  $R$  satisfies the equation:

$$\begin{aligned} & \mathcal{T}(R) + (B \cdot \nabla_u R) - \frac{\sigma^2}{2} \Delta_u R = 0 \text{ on } Q_T, \\ & R(0, x, u) = S(f_1)(0) - S(f_2)(0) = 0 \text{ for } (x, u) \in \mathcal{D} \times \mathbb{R}^d, \\ & \gamma^+(R)(t, x, u) = (\gamma^+(f_1) - \gamma^+(f_2))(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)) \text{ on } \Sigma_T^-. \end{aligned}$$

Using Lemma 3.7 and following the proof of the uniqueness for (3.24), one has:

$$\min(1, \sigma^2) \|R\|_{V_1(\omega, Q_t)}^2 + \|\gamma^+(R)\|_{L^2(\omega, \Sigma_t^+)}^2 \leq \|\gamma^-(R)\|_{L^2(\omega, \Sigma_t^-)}^2 + C \int_0^t \|R\|_{V_1(\omega, Q_s)}^2 ds \quad (3.27)$$

for  $C := \frac{\sigma^2}{2} (2\alpha (\frac{\alpha}{2} - 1) + \alpha d) + \frac{\alpha}{2} \|B\|_{L^\infty((0, T) \times \mathcal{D}; \mathbb{R}^d)}$ . Applying Gronwall's Lemma, it follows that

$$\|S(f_2) - S(f_1)\|_E^2 \leq C \|\gamma^-(S(f_2)) - \gamma^-(S(f_1))\|_{L^2(\omega, \Sigma_T^-)}^2 = C \|\gamma^+(f_2) - \gamma^+(f_1)\|_{L^2(\omega, \Sigma_T^+)}^2,$$

which enable us to deduce

$$\|S(f_2) - S(f_1)\|_E^2 \leq C \|f_2 - f_1\|_E^2, \quad (3.28)$$

and thus that  $S$  is Lipschitz-continuous.

**Proof of (3).** Assume that  $f \in E$  is such that

$$\underline{P}(t, u) \leq \gamma^+(f)(t, x, u) \leq \overline{P}(t, u), \lambda_{\Sigma} \text{ for a.e. } (t, x, u) \in \Sigma_T^+. \quad (3.29)$$

By assumption on  $\underline{P}_0$  and  $\overline{P}_0$ , Remark 3.2 implies that

$$\begin{aligned} \underline{P}(t, u) &= \underline{P}(t, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), \\ \overline{P}(t, u) &= \overline{P}(t, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)). \end{aligned}$$

Hence (3.29) is equivalent to

$$\underline{P}(t, u) \leq \gamma^+(f)(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)) \leq \overline{P}(t, u), \quad \lambda_{\Sigma}\text{-a.e. on } \Sigma_T^-, \quad (3.30)$$

Applying Proposition 3.8, it follows that

$$\begin{aligned} \underline{P} &\leq S(f) \leq \overline{P}, \quad \text{a.e. on } Q_T, \text{ and} \\ \underline{P} &\leq \gamma^+(S(f)) \leq \overline{P}, \quad \lambda_{\Sigma}\text{-a.e. on } \Sigma_T^+. \end{aligned}$$

Moreover, according to Lemma 3.11, (3.30) yields

$$\underline{P}(t, u) \leq \gamma^-(S(f))(t, x, u) \leq \overline{P}(t, u), \quad \lambda_{\Sigma} \text{ for a.e. } (t, x, u) \in \Sigma_T^-.$$

We thus formulate the Maxwellian bounds (3).

**Proof of (2).** Let  $f_1, f_2$  be such that  $f_1 \leq f_2$  on  $E$ . The difference  $S(f_2) - S(f_1)$  is then a weak solution to the linear Vlasov–Fokker–Planck equation (3.11) for  $\rho_0 = 0$  and  $g(t, x, u) = (\gamma^+(f_2) - \gamma^+(f_1))(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x))$ ; namely

$$\begin{cases} \mathcal{T}(S(f_2) - S(f_1)) + (B \cdot \nabla_u(S(f_2) - S(f_1))) - \frac{\sigma^2}{2} \Delta_u(S(f_2) - S(f_1)) = 0 \text{ on } Q_T, \\ S(f_2)(0, x, u) - S(f_1)(0, x, u) = 0 \text{ on } \mathcal{D} \times \mathbb{R}^d, \\ (S(f_2) - S(f_1))(t, x, u) = (\gamma^+(f_2) - \gamma^+(f_1))(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)) \text{ on } \Sigma_T^-. \end{cases}$$

Therefore, applying Lemma 3.7, we obtain that  $S(f_2) - S(f_1) = S(f_2 - f_1) \geq 0$  and that  $\gamma^+(S(f_2) - S(f_1)) = \gamma^+(S(f_2 - f_1)) \geq 0$ . Moreover, by identifying  $\gamma^+(S(f_2) - S(f_1))$  to  $\gamma^+(S(f_2)) - \gamma^+(S(f_1))$ , and we conclude on the nondecreasing of  $S$  on  $E$  since  $\gamma^-(S(f_2)) - \gamma^-(S(f_1))(t, x, u) = (\gamma^+(f_2) - \gamma^+(f_1))(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)) \geq 0$ .  $\square$

**Step 2: Introduction of the nonlinear drift (and end of the proof of Theorem 3.3).** Hereafter we end the proof of Theorem 3.3 by introducing the non-linearity  $B[\cdot; \cdot]$  in the equation (3.24). To this aim, we consider  $(\underline{P}, \overline{P})$ , a couple of Maxwellian distributions with parameters  $(\underline{a}, \underline{\mu}, \underline{P}_0)$  and  $(\overline{a}, \overline{\mu}, \overline{P}_0)$  such that

$$\begin{aligned} 2\underline{\mu} &> 1, \quad 2\overline{\mu} > 1, \\ \underline{a} &\leq \frac{-\underline{\mu}}{2\sigma^2(\underline{\mu} - 1)} \|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}^2, \quad \overline{a} \geq \frac{\overline{\mu}}{2\sigma^2(1 - \overline{\mu})} \|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}^2. \end{aligned} \quad (3.31)$$

We also consider the sequence  $\{\rho^{(n)}\}_{n \in \mathbb{N}}$  defined by:

- $\rho^{(0)} = \rho_0$  on  $Q_T$ ;
- For all  $n \geq 1$ , given  $B(t, x) = B[x; \rho^{(n-1)}(t)]$  with  $B : \mathcal{D} \times L^1(\mathcal{D} \times \mathbb{R}^d) \rightarrow \mathbb{R}^d$  defined as in (1.2), and since, under  $(H_{\text{VFP}})\text{-(i)}$ ,  $|B(t, x)| \leq \|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}$ , for a.e.  $(t, x) \in (0, T) \times \mathcal{D}$ , we define  $\rho^{(n)}$  as the unique solution in  $V_1(\omega, Q_T)$  of

$$\begin{aligned} \mathcal{T}(\rho^{(n)}) &= \frac{\sigma^2}{2} \Delta_u \rho^{(n)} - \left( \nabla_u \cdot B \rho^{(n)} \right) - (B[\cdot; \rho^{(n-1)}] \cdot \nabla_u \rho^{(n)}) \text{ on } \mathcal{H}'(Q_T), \\ \gamma^-(\rho^{(n)})(t, x, u) &= \gamma^+(\rho^{(n)})(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)) \text{ in } \Sigma_T^-, \\ \rho^{(n)}(0, x, u) &= \rho_0(x, u) \text{ in } \mathcal{D} \times \mathbb{R}^d. \end{aligned} \quad (3.32)$$

According to Proposition 3.13, it holds: for all  $n \geq 1$ ,

$$\begin{aligned} \underline{P} &\leq \rho^{(n)} \leq \overline{P} \text{ a.e. on } Q_T, \\ \underline{P} &\leq \gamma^\pm(\rho^{(n)}) \leq \overline{P} \quad \lambda_{\Sigma}\text{-a.e. on } \Sigma_T^\pm, \end{aligned} \quad (3.33)$$

so that  $\rho^{(n)}$  is positive on  $Q_T$  and

$$B[x; \rho^{(n-1)}(t)] = \frac{\int_{\mathbb{R}^d} b(v) \rho^{(n-1)}(t, x, v) dv}{\int_{\mathbb{R}^d} \rho^{(n-1)}(t, x, v) dv}, \text{ for a.e. } (t, x) \in (0, T) \times \mathcal{D}.$$

**Proposition 3.14.** *Assume  $(H_{\text{VFP}})$ . Let  $(\underline{P}, \overline{P})$  be a couple of Maxwellian distributions with respective parameters  $(\underline{a}, \underline{\mu}, \underline{P}_0)$  and  $(\overline{a}, \overline{\mu}, \overline{P}_0)$  verifying (3.31). Then the sequence  $\{\rho^{(n)}\}_{n \in \mathbb{N}}$  converges in  $V_1(\omega, Q_T)$  to a weak solution  $\rho$  to the non-linear equation (3.1a)–(3.1c). Moreover this solution verifies*

$$\begin{aligned} \underline{P} &\leq \rho \leq \overline{P}, \text{ a.e. on } Q_T, \\ \underline{P} &\leq \gamma^\pm(\rho) \leq \overline{P}, \text{ } \lambda_\Sigma\text{-a.e. on } \Sigma_T^\pm. \end{aligned}$$

*Proof.* As a preliminary step, let us remark that, when we replace the Dirichlet condition (3.8c) by the specular boundary condition in the linear Vlasov-Fokker-Planck equation (3.8) and since

$$\int_{\Sigma_t^-} (u \cdot n_{\mathcal{D}}(x)) \omega(u) |\gamma^+(f)(s, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x))| d\lambda_{\Sigma_t}(s, x, u) = \|\gamma^-(f)\|_{L^2(\omega, \Sigma_t^-)}^2,$$

then the  $L^2$ -estimate (3.9) in Lemma 3.7 rewrites as

**Corollary 3.15.** *Given  $B \in L^\infty((0, T) \times \mathcal{D})$  and  $f_0 \in L^2(\omega, \mathcal{D} \times \mathbb{R}^d)$ , let  $f \in V_1(\omega, Q_T)$  be the solution to*

$$\mathcal{T}(f) = \frac{\sigma^2}{2} \Delta_u f - (\nabla_u \cdot B f) + g, \text{ in } \mathcal{H}'(Q_T), \quad (3.34a)$$

$$f(0, x, u) = f_0(x, u), \text{ on } \mathcal{D} \times \mathbb{R}^d, \quad (3.34b)$$

$$\gamma^-(f)(t, x, u) = \gamma^+(f)(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), \text{ on } \Sigma_T^-, \quad (3.34c)$$

then, for all  $t \in (0, T]$ ,

$$\|f(t)\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 + \sigma^2 \|\nabla_u f\|_{L^2(\omega, Q_t)}^2 = \|f_0\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 + \int_{Q_T} \left( \frac{\sigma^2}{2} \Delta_u \omega + (\nabla_u \omega \cdot B) \right) |f|^2 + 2 \int_{Q_t} \omega g f. \quad (3.35)$$

We now start the proof by establishing a uniform estimation for  $\{\rho^{(n)}\}_{n \in \mathbb{N}}$  of the form:

$$\sup_{n \geq 1} \|\rho^{(n)}\|_{V_1(\omega, Q_T)}^2 \leq K \left( \|\rho_0\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 + \|\overline{P}\|_{L^2(\omega, Q_T)}^2 \right), \quad (3.36)$$

for some constant  $K > 0$ . Using the energy estimate (3.35) (using  $g = B[\cdot; \rho^{(n-1)}] \nabla_u \rho^{(n)}$ ), we obtain that: for all  $t \in (0, T]$ ,

$$\begin{aligned} &\|\rho^{(n)}(t)\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 + \sigma^2 \|\nabla_u \rho^{(n)}\|_{L^2(\omega, Q_t)}^2 \\ &\leq \|\rho_0\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 + \frac{\sigma^2 C_1}{2} \int_{Q_t} \|\rho^{(n)}(s)\|_{L^2(\omega, (\mathcal{D} \times \mathbb{R}^d))}^2 ds + \frac{\alpha}{2} \int_{Q_t} \omega \left| \rho^{(n)} \right|^2 \left| B[\cdot; \rho^{(n-1)}] \right|, \end{aligned}$$

for  $C_1 := \left( 2\alpha \left( \frac{\alpha}{2} - 1 \right) + \alpha d \right)$ . Using the bound  $B[\cdot; \rho^{(n-1)}] \leq \|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}$ , we deduce that, for all  $t \in (0, T]$ ,

$$\min(1, \sigma^2) \|\rho^{(n)}\|_{V_1(\omega, Q_t)}^2 \leq \|\rho_0\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 + C_2 \int_0^t \|\rho^{(n)}\|_{L^2(\omega, Q_s)}^2 ds,$$

for  $C_2 := \frac{\sigma^2 C_1}{2} + \frac{\alpha \|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}}{2}$ . Using the Maxwellian upper-bound (3.33), we obtain (3.36).

We are now interested in the strong-convergence of  $\{\rho^{(n)}\}_{n \in \mathbb{N}}$ . Since  $V_1(\omega, Q_T)$  equipped with the norm  $\|\cdot\|_{V_1(\omega, Q_T)}$  is a Banach space, it is sufficient to establish that  $\{\rho^{(n)}\}_{n \in \mathbb{N}}$  is a Cauchy sequence in  $V_1(\omega, Q_T)$ . Take  $t \in (0, T]$ . For  $n, m > 1$ , the functions

$$\begin{aligned} R^{(n, n+m)} &:= \rho^{(n+m)} - \rho^{(n)} \text{ on } Q_T, \\ \gamma^\pm(R^{(n, n+m)}) &:= \gamma^\pm(\rho^{(n+m)}) - \gamma^\pm(\rho^{(n)}), \text{ on } \Sigma_T^\pm, \end{aligned}$$

satisfy

$$\begin{cases} \mathcal{T}(R^{(n, n+m)}) - \frac{\sigma^2}{2} \Delta_u R^{(n, n+m)} = (B[\cdot; \rho^{(n)}] \cdot \nabla_u \rho^{(n)}) - (B[\cdot; \rho^{(n+m)}] \cdot \nabla_u \rho^{(n+m)}) \text{ in } \mathcal{H}'(Q_T), \\ R^{(n, n+m)}(0, x, u) = 0 \text{ on } \mathcal{D} \times \mathbb{R}^d, \\ \gamma^-(R^{(n, n+m)})(t, x, u) = \gamma^+(R^{(n, n+m)})(t, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)) \text{ on } \Sigma_T^-. \end{cases}$$

Hence, according to (3.35), it follows that

$$\begin{aligned} & \int_{\mathcal{D} \times \mathbb{R}^d} \omega(u) \left| R^{(n, n+m)}(t, x, u) \right|^2 dx du + \sigma^2 \int_{Q_t} \omega \left| \nabla_u R^{(n, n+m)} \right|^2 \\ & \leq \frac{\sigma^2}{2} \int_{Q_t} \Delta \omega(u) \left| R^{(n, n+m)} \right|^2 + 2 \int_{Q_T} \omega R^{(n, n+m)} \left( \nabla_u \cdot (B[\cdot; \rho^{(n-1)}] \rho^{(n)} - \rho^{(n+m)} B[\cdot; \rho^{(n+m-1)}]) \right) \\ & \leq \left( \frac{(C_1 + 1)\sigma^2}{2} + \alpha \right) \int_0^t \|R^{(n, n+m)}\|_{V^1(\omega, Q_s)}^2 ds + \left( \frac{1}{2\sigma^2} + \frac{\alpha}{2} \right) \int_{Q_t} \omega \left| \rho^{(n+m)} B[\cdot; \rho^{(n+m-1)}] - \rho^{(n)} B[\cdot; \rho^{(n-1)}] \right|^2. \end{aligned}$$

Let us now observe that

$$\begin{aligned} & \int_{Q_t} \omega \left| \rho^{(n+m)} B[\cdot; \rho^{(n+m-1)}] - \rho^{(n)} B[\cdot; \rho^{(n-1)}] \right|^2 \\ & \leq \frac{1}{2} \int_{Q_t} \omega \left| \rho^{(n+m)} - \rho^{(n)} \right|^2 \left| B[\cdot; \rho^{(n+m-1)}] + B[\cdot; \rho^{(n-1)}] \right|^2 + \frac{1}{2} \int_{Q_t} \omega \left| \rho^{(n+m)} + \rho^{(n)} \right|^2 \left| B[\cdot; \rho^{(n+m-1)}] - B[\cdot; \rho^{(n-1)}] \right|^2. \end{aligned}$$

For the first term, we have

$$\int_{Q_t} \omega \left| \rho^{(n+m)} - \rho^{(n)} \right|^2 \left| B[\cdot; \rho^{(n+m-1)}] + B[\cdot; \rho^{(n-1)}] \right|^2 \leq 2\|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}^2 \int_{Q_t} \omega \left| \rho^{(n+m)} - \rho^{(n)} \right|^2. \quad (3.37)$$

For the second term, let us set

$$\underline{M} := \operatorname{ess\,inf}_{t \in (0, T)} \int_{\mathbb{R}^d} \underline{P}(t, u) du, \quad \overline{M} := \operatorname{ess\,sup}_{t \in (0, T)} \int_{\mathbb{R}^d} \omega(u) \left| \overline{P}(t, u) \right|^2 du,$$

According to the definition of  $B[\cdot; \cdot]$  in (1.2),

$$\begin{aligned} & \left| B[x; \rho^{(n+m-1)}(s)] - B[x; \rho^{(n-1)}(s)] \right|^2 \\ & = \left| \frac{\int_{\mathbb{R}^d} b(v) \rho^{(n+m-1)}(s, x, v) dv \int_{\mathbb{R}^d} \rho^{(n)}(s, x, v) dv - \int_{\mathbb{R}^d} b(v) \rho^{(n-1)}(s, x, v) dv \int_{\mathbb{R}^d} \rho^{(n+m-1)}(s, x, v) dv}{\int_{\mathbb{R}^d} \rho^{(n+m-1)}(s, x, v) dv \int_{\mathbb{R}^d} \rho^{(n-1)}(s, x, v) dv} \right|^2 \\ & \leq \frac{2 \left| \int_{\mathbb{R}^d} b(v) (\rho^{(n+m-1)} - \rho^{(n)}) (s, x, v) dv \right|^2}{\left| \int_{\mathbb{R}^d} \rho^{(n+m-1)}(s, x, v) dv \right|^2} + \frac{2 \left| \int_{\mathbb{R}^d} (\rho^{(n+m-1)} - \rho^{(n-1)}) (s, x, v) dv \right|^2 \left| \int_{\mathbb{R}^d} b(v) \rho^{(n-1)}(s, x, v) dv \right|^2}{\left| \int_{\mathbb{R}^d} \rho^{(n+m-1)}(s, x, v) dv \int_{\mathbb{R}^d} \rho^{(n-1)}(s, x, v) dv \right|^2} \\ & \leq \frac{4\|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}^2 \left( \int_{\mathbb{R}^d} \left| \rho^{(n+m-1)} - \rho^{(n-1)} \right| (s, x, v) dv \right)^2}{(\underline{M})^2}. \end{aligned}$$

Since  $\overline{w} := \int_{\mathbb{R}^d} \frac{1}{\omega(v)} dv$  is a finite constant (see Lemma 3.4), Cauchy-Schwarz's inequality induces

$$\left( \int_{\mathbb{R}^d} |\rho^{n+m-1}(s, x, v) - \rho^{n-1}(s, x, v)| dv \right)^2 \leq \overline{w} \int_{\mathbb{R}^d} \omega(v) |\rho^{n+m-1} - \rho^{n-1}|^2(s, x, v) dv.$$

Therefore,

$$|B[x; \rho^{n+m-1}(s)] - B[x; \rho^{n-1}(s)]|^2 \leq \frac{4\overline{w}\|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}^2}{(\underline{M})^2} \int_{\mathbb{R}^d} \omega(v) |(\rho^{n+m-1} - \rho^{n-1})(s, x, v)|^2 dv,$$

from which it follows

$$\begin{aligned} & \int_{Q_t} \omega |\rho^{n+m} + \rho^n|^2 |B[\cdot; \rho^{n+m-1}] - B[\cdot; \rho^{n-1}]|^2 \\ &= \int_{(0,t) \times \mathcal{D}} \left( \int_{\mathbb{R}^d} \omega(u) |\rho^{n+m}(s, x, u) + \rho^n(s, x, u)|^2 du \right) |B[\cdot; \rho^{n+m-1}] - B[\cdot; \rho^{n-1}]|^2(s, x) ds dx \\ &\leq 2\overline{M} \int_{(0,t) \times \mathcal{D}} |B[\cdot; \rho^{n+m-1}] - B[\cdot; \rho^{n-1}]|^2(s, x) ds dx \\ &\leq \frac{4\overline{w}\overline{M}\|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}^2}{(\underline{M})^2} \int_{Q_t} \omega |\rho^{n+m-1} - \rho^{n-1}|^2. \end{aligned}$$

Combining this inequality with (3.37), we obtain that

$$\begin{aligned} & \int_{Q_t} \omega \left| \rho^{(n+m)} B[\cdot; \rho^{(n+m-1)}] - \rho^{(n)} B[\cdot; \rho^{(n-1)}] \right|^2 \\ &\leq \|b\|_{L^\infty(\mathbb{R}^d); \mathbb{R}^d}^2 \int_{Q_t} \omega \left| R^{(n,n+m)} \right|^2 + \frac{2\overline{w}\overline{M}\|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}^2}{(\underline{M})^2} \int_{Q_t} \omega \left| R^{(n-1,n+m-1)} \right|^2. \end{aligned}$$

Hence for all  $t$  in  $(0, T]$ ,

$$\min(1, \sigma^2) \|R^{(n,n+m)}\|_{V_1(\omega, Q_t)}^2 \leq 2C_3 \int_0^t \|R^{(n,n+m)}\|_{V_1(\omega, Q_s)}^2 ds + 2C_4 \int_0^t \|R^{(n+m-1,n-1)}\|_{V_1(\omega, Q_s)}^2 ds,$$

with

$$C_3 = \left( \frac{(C_1 + 1)\sigma^2}{2} + \alpha + \|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}^2 \right), \quad C_4 = 2 \left( \frac{1}{2\sigma^2} + \frac{\alpha}{2} \right) \|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}^2 \left( 1 + \frac{4\overline{M}\overline{w}}{(\underline{M})^2} \right).$$

Therefore, applying Gronwall's Lemma, we deduce that, for  $C_5 = \frac{2C_3}{\min(1, \sigma^2)}$ ,

$$\|\rho^{(n+m)} - \rho^{(n)}\|_{V_1(\omega, Q_t)}^2 \leq C_4 \int_0^t (1 + \exp\{C_5 s\}) \|\rho^{(n+m-1)} - \rho^{(n-1)}\|_{V_1(\omega, Q_s)}^2 ds.$$

Iterating  $n$  times this inequality, we obtain

$$\begin{aligned} & \|\rho^{(n+m)} - \rho^{(n)}\|_{V_1(\omega, Q_T)}^2 \\ &\leq \int_0^T C_4 (1 + \exp\{C_5 t\}) \int_0^{t_{n-1}} \cdots \int_0^{t_2} C_4 (1 + \exp\{C_5 t_0\}) \|\rho^{(m+1)} - \rho^{(1)}\|_{V_1(\omega, Q_{t_0})}^2 dt_0 \cdots dt_{n-1} dt_n \\ &\leq \frac{2K(C_T)^n}{n!} \|\rho^{(m+1)} - \rho^{(1)}\|_{V_1(\omega, Q_T)}^2, \end{aligned}$$

for  $C_T := C_4 \int_0^T (1 + \exp\{C_5 t\}) dt$ . Using the estimation (3.36), it follows that

$$\sup_{m \in \mathbb{N}} \|\rho^{(n+m)} - \rho^{(n)}\|_{V_1(\omega, Q_T)}^2 \leq \frac{2K(C_T)^n}{n!} \left( \|\rho_0\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 + \overline{M} \right).$$

Since  $\sum_{n \in \mathbb{N}} \frac{(C_T)^n}{n!} = \exp \{C_T\} < +\infty$ , the coefficients  $\frac{(C_T)^n}{n!}$  tend to 0 as  $n$  tends to infinity. Therefore  $\{\rho^{(n)}\}_{n \in \mathbb{N}}$  is a Cauchy sequence of  $V_1(\omega, Q_T)$ .

Let us denote by  $\rho$ , the limit of  $\{\rho^{(n)}; n \geq 1\}$  in  $V_1(\omega, Q_T)$ . According to (3.33), we have

$$\underline{P} \leq \rho \leq \overline{P}, \text{ a.e. on } Q_T.$$

We thus check that  $\rho$  is a weak solution to (3.1a)–(3.1c). Since  $\{\rho^n\}$  tends to  $\rho$  in  $V_1(\omega, Q_T)$ , and so in  $L^2((0, T) \times \mathcal{D}; H^1(\omega, \mathbb{R}^d))$ , we can consider a subsequence still denoted by  $\{\rho^{(n)}; n \geq 1\}$  such that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \rho^{(n)}(t, x, u) &= \rho(t, x, u), \text{ for a.e. } (t, x, u) \in Q_T, \\ \lim_{n \rightarrow +\infty} \nabla_u \rho^{(n)}(t, x, u) &= \nabla_u \rho(t, x, u), \text{ for a.e. } (t, x, u) \in Q_T, \end{aligned}$$

and

$$\sup_{n \geq 1} \left( \left| \rho^{(n)}(t, x, u) \right| + \left| \nabla_u \rho^{(n)}(t, x, u) \right| \right) \in L^2(\omega, Q_T).$$

Therefore, by dominated convergence, we deduce the convergence of the coefficients

$$\lim_{n \rightarrow +\infty} B[t, x; \rho^{(n)}] = B[t, x; \rho], \text{ a.e. on } (0, T) \times \mathbb{R}^d.$$

We further observe that  $\psi \in \mathcal{C}_c^\infty(Q_T)$ ,

$$\begin{aligned} \left| \int_{Q_T} \mathcal{T}(\psi) \sqrt{\omega} \rho \right| &\leq \limsup_{n \rightarrow +\infty} \left| \int_{Q_T} \mathcal{T}(\psi) \sqrt{\omega} \rho^{(n)} \right| \\ &\leq \limsup_{n \rightarrow +\infty} \left| \int_{Q_T} \sqrt{\omega} \psi \left( B[\cdot; \rho^{(n-1)}] \cdot \nabla_u \rho^{(n)} \right) - \frac{\sigma^2}{2} \left( \nabla_u(\sqrt{\omega} \psi) \cdot \nabla_u \rho^{(n)} \right) + \beta(u \cdot \nabla_u \sqrt{\omega}) \psi \rho^{(n)} \right| \\ &\leq \left( \|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)} + \frac{\sigma^2}{2} \left( 1 + \frac{\alpha}{2} \right) \right) \|\psi\|_{\mathcal{H}(Q_T)} \limsup_{n \rightarrow +\infty} \|\rho^n\|_{V_1(\omega, Q_T)}, \end{aligned}$$

by using the estimations  $(u \cdot \sqrt{\omega}(u))$  given in Lemma 3.4. Owing to (3.36), the right member of the inequality is uniformly bounded, and so  $\mathcal{T}(\sqrt{\omega} \rho) \in \mathcal{H}'(Q_T)$  and, from (3.32), that: for all  $t \in [0, T]$ , and, for all  $\psi \in \mathcal{C}_c^\infty(\overline{Q_t})$  vanishing in  $\partial \mathcal{D}$ ,

$$\mathcal{T}(\rho) + (B[\cdot; \rho] \cdot \nabla_u \rho) - \frac{\sigma^2}{2} \Delta_u \rho = 0$$

According to Lemma 3.6,  $\rho$  thus admits traces functions along the frontier  $\Sigma_T^\pm$ , which belongs to  $L^2(\omega, \Sigma_T^\pm)$ . It thus remains to check the specular boundary condition and the Maxwellian bounds hold for  $\rho$ ; that is

$$\begin{aligned} \gamma^-(\rho)(s, x, u) &= \gamma^+(\rho)(s, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), \quad \lambda_\Sigma \text{-a.e. on } \Sigma_T^-, \\ \underline{P} &\leq \gamma^\pm(\rho) \leq \overline{P}, \quad \lambda_\Sigma \text{-a.e. on } \Sigma_T^\pm. \end{aligned} \tag{3.38}$$

On the other hand, for all  $\psi \in \mathcal{C}_c^\infty(\overline{Q_T})$ , we have

$$\begin{aligned} &\int_{Q_T} \mathcal{T}(\psi) (\rho - \rho^{(n)}) + \int_{Q_T} \psi \left( (B[\cdot; \rho] \cdot \nabla_u \rho) - (B[\cdot; \rho^{(n-1)}] \cdot \nabla_u \rho^{(n)}) \right) \\ &+ \frac{\sigma^2}{2} \int_{Q_T} \left( \nabla_u \psi \cdot \nabla_u (\rho - \rho^{(n)}) \right) = - \int_{\mathcal{D} \times \mathbb{R}^d} \psi(T, x, u) \left( \rho(T, x, u) - \rho^{(n)}(T, x, u) \right) dx du \\ &- \int_{\Sigma_T^+} (u \cdot n_{\mathcal{D}}(x)) \psi(s, x, u) \left( \gamma^+(\rho) - \gamma^+(\rho^{(n)}) \right) (s, x, u) d\lambda_\Sigma(s, x, u) \\ &- \int_{\Sigma_T^-} (u \cdot n_{\mathcal{D}}(x)) \psi(s, x, u) \left( \gamma^-(\rho)(s, x, u) - \gamma^-(\rho^{(n)})(s, x, u) \right) d\lambda_\Sigma(s, x, u). \end{aligned}$$

Hence, for all  $\psi \in \mathcal{C}_c^\infty(\overline{Q_T})$  vanishing on  $\{T\} \times \mathcal{D} \times \mathbb{R}^d$  and  $\Sigma_T^+$ , we have

$$\lim_{n \rightarrow +\infty} \int_{\Sigma_T^-} (u \cdot n_{\mathcal{D}}(x)) \psi(s, x, u) \left( \gamma^-(\rho)(s, x, u) - \gamma^-(\rho^{(n)})(s, x, u) \right) d\lambda_\Sigma(s, x, u) = 0.$$

It follows that

$$\lim_{n \rightarrow +\infty} \|\gamma^-(\rho) - \gamma^-(\rho^{(n)})\|_{L^2(\Sigma_T^-)} = 0.$$

Since, for all  $n \geq 1$ ,  $\rho^{(n)}$  satisfies the specular boundary condition, and by (3.33), we deduce that (3.38). Then we conclude that  $\rho$ , the limit of  $\{\rho^{(n)}\}_{n \in \mathbb{N}}$ , is the unique weak solution to the McKean–Vlasov–Fokker–Planck equation (3.1a)–(3.1c).  $\square$

## 4 Wellposedness for the nonlinear Lagrangian model with specular boundary condition

### 4.1 Construction of a solution

In this section we construct a solution to the Lagrangian model with specular boundary conditions (1.1). First, using Theorem 2.1, let us denote by  $\mathbb{P}$  the probability measure defined on the sample space  $\mathcal{T}$  such that, under  $\mathbb{P}$ , the canonical processes  $(x(t), u(t))$  of  $\mathcal{T}$  satisfies

$$\begin{cases} x(t) = x(0) + \int_0^t u(s) ds, \\ u(t) = u(0) + \sigma \tilde{w}(t) - \sum_{0 < s \leq t} 2(u(s^-) \cdot n_{\mathcal{D}}(x(s))) n_{\mathcal{D}}(x(s)) \mathbb{1}_{\{x(s) \in \partial \mathcal{D}\}}, \end{cases}$$

where  $(\tilde{w}(t))$  is a  $\mathbb{R}^d$ -Brownian motion. Next, starting from the solution  $\rho^{\text{FP}} \in V_1(\omega, Q_T)$  to the conditional McKean–Vlasov–Fokker–Planck equation related to (1.1) that we constructed in Theorem 3.3, we introduce the probability measure  $\mathbb{Q}$  defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left\{ \frac{1}{\sigma} \int_0^T B[x(t); \rho^{\text{FP}}(t)] dw(t) - \frac{1}{2\sigma^2} \int_0^T |B[x(t); \rho^{\text{FP}}(t)]|^2 dt \right\}. \quad (4.1)$$

Then, according to Girsanov theorem,  $(x(t), u(t))$  satisfies under  $\mathbb{Q}$  the equation of the confined Langevin model with the additional drift  $t \mapsto \int_0^t B[x(s); \rho^g(s)] ds$ ; namely,  $\mathbb{Q}$ -a.s.,

$$\begin{cases} x(t) = x(0) + \int_0^t u(s) ds, \\ u(t) = u(0) + \int_0^t B[x(s); \rho^{\text{FP}}(s)] ds + \sigma w(t) - \sum_{0 < s \leq t} 2(u(s^-) \cdot n_{\mathcal{D}}(x(s))) n_{\mathcal{D}}(x(s)) \mathbb{1}_{\{x(s) \in \partial \mathcal{D}\}}, \end{cases}$$

where  $(w(t) := \tilde{w}(t) - \int_0^t B[x(s); \rho^{\text{FP}}(s)] ds)$  is a  $\mathbb{R}^d$ -Brownian motion and  $\mathbb{Q}(x(0) \in dx, u(0) \in du) = \rho_0(x, u) dx du$ . To prove that  $(\mathbb{Q}, x(t), u(t))$  is indeed a solution in law to (1.1), we check that the time-marginals of  $\mu(t) := \mathbb{Q} \circ (x(t), u(t))^{-1}$  have the density functions  $\rho^{\text{FP}}(t)$  so that  $B[x(t); \rho^{\text{FP}}(t)]$  is equal to  $\mathbb{E}_{\mathbb{Q}}[b(u(t))/x(t)]$ . To show the correspondance between  $\mu(t)$  and  $\rho^{\text{FP}}$ , we use the following mild formulation:  $\rho \in \mathcal{C}([0, T]; L^2(\mathcal{D} \times \mathbb{R}^d))$  will be said a mild solution to the linear Vlasov–Fokker–Planck equation with specular boundary condition:

$$\begin{cases} \partial_t \rho(t) + (u \cdot \nabla_x \rho(t)) + (B[x; \rho^{\text{FP}}(t)] \cdot \nabla_u \rho(t)) - \frac{\sigma^2}{2} \Delta_u \rho(t) = 0 \text{ on } Q_T, \\ \rho(0, x, u) = \rho_0(x, u) \text{ on } \mathcal{D} \times \mathbb{R}^d, \\ \rho(t, x, u) = \rho(t, x, u - 2(u \cdot n_{\mathcal{D}}(x)) n_{\mathcal{D}}(x)) \text{ on } \Sigma_T^+. \end{cases} \quad (4.2)$$

if and only if  $\rho$  verifies the mild equation: for all  $t \in (0, T]$ ,  $\psi \in \mathcal{C}_c^\infty(\mathcal{D} \times \mathbb{R}^d)$ ,

$$\langle \psi, \rho(t) \rangle = \langle \Gamma^\psi(t), \rho_0 \rangle + \int_0^t \langle \nabla_u \Gamma^\psi(t-s), B[\cdot; \rho^{\text{FP}}(s)] \rho(s) \rangle ds, \quad \forall \psi \in \mathcal{C}_c^\infty(\mathcal{D} \times \mathbb{R}^d), \quad (4.3)$$

where  $\langle \cdot, \cdot \rangle$  stands for the inner product in  $L^2(\mathcal{D} \times \mathbb{R}^d)$  and  $\Gamma^\psi(t, x, u)$  is given as in (2.16). According to Proposition 2.5, for all  $\psi \in \mathcal{C}_c^\infty(\mathcal{D} \times \mathbb{R}^d)$ ,  $\Gamma^\psi$  belongs to  $\mathcal{C}([0, T]; L^2(\mathcal{D} \times \mathbb{R}^d)) \cap L^2([0, T] \times \mathcal{D}; H^1(\mathbb{R}^d))$  hence (4.3) is well defined. Furthermore, we have

**Proposition 4.1.** *There exists at most one solution in  $\mathcal{C}([0, T]; L^2(\mathcal{D} \times \mathbb{R}^d))$  to the linear mild equation (4.3).*

*Proof.* Let  $\rho_1, \rho_2 \in \mathcal{C}([0, T]; L^2(\mathcal{D} \times \mathbb{R}^d))$  be two mild solutions to (4.3). Then, for all  $t \in [0, T]$ ,

$$\begin{aligned} \|\rho_1(t) - \rho_2(t)\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 &= \left( \sup_{\psi \in \mathcal{C}_c^\infty(\mathcal{D} \times \mathbb{R}^d); \|\psi\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}=1} \langle \psi, (\rho_1 - \rho_2)(t) \rangle \right)^2 \\ &\leq \sup_{\psi \in \mathcal{C}_c^\infty(\mathcal{D} \times \mathbb{R}^d); \|\psi\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}=1} \left| \int_{Q_t} (\nabla_u \Gamma^\psi(t-s, x, u), B[x; \rho^{\text{FP}}(s)]) (\rho_1 - \rho_2)(s, x, u) ds dx du \right|^2 \\ &\leq \|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}^2 \left( \sup_{\psi \in L^2(\mathcal{D} \times \mathbb{R}^d); \|\psi\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}=1} \|\nabla_u \Gamma^\psi\|_{L^2(Q_t)}^2 \right) \int_0^t \|\rho_1(s) - \rho_2(s)\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 ds. \end{aligned}$$

Using the estimate (2.23) on  $\|\nabla_u \Gamma^\psi\|_{L^2(\omega, Q_T)}$  in Corollary 2.7, it follows that

$$\|\rho_1(t) - \rho_2(t)\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 \leq \frac{\|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}^2}{\sigma^2} \int_0^t \|\rho_1(s) - \rho_2(s)\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 ds.$$

By Gronwall's lemma, we conclude on the uniqueness of solutions to (4.3).  $\square$

**Proposition 4.2.** (i) *The weak solution  $(\rho^{\text{FP}}(t))$  constructed in Theorem 3.3 is a mild solution to (4.3).*  
(ii) *For all  $t$ , the time marginal density  $\rho(t)$  of  $\mathbb{Q}$  defined in (4.1) is a mild solution to (4.3).*

Combining Proposition 4.2 and Proposition 4.1 we conclude on the equality  $\rho(t) = \rho^{\text{FP}}(t)$  and that  $\mathbb{Q}$  is actually a solution in law to (1.1) in  $\Pi_\omega$ .

*Proof of Proposition 4.2.* According to Theorem 3.3,  $\rho^{\text{FP}}$  satisfies: for all  $t \in [0, T]$ ,  $\Psi \in \mathcal{C}_c^\infty(\overline{Q_t})$ ,

$$\begin{aligned} &\int_{Q_t} \left( \rho^{\text{FP}}(s, x, u) \left( \partial_s \Psi + (u \cdot \nabla_x \Psi) + \frac{\sigma^2}{2} \Delta_u \Psi \right) (s, x, u) + (B[x; \rho^{\text{FP}}(s)] \cdot \nabla_u \Psi(s, x, u)) \rho^{\text{FP}}(s, x, u) \right) ds dx du \\ &= - \int_{\mathcal{D} \times \mathbb{R}^d} \rho^{\text{FP}}(t, x, u) \Psi(t, x, u) dx du + \int_{\mathcal{D} \times \mathbb{R}^d} \rho_0(x, u) \Psi(0, x, u) dx du \\ &\quad - \int_{\Sigma_t^+} (u \cdot n_{\mathcal{D}}(x)) \gamma^+(\rho^{\text{FP}})(s, x, u) (\Psi(s, x, u) - \Psi(s, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x))) d\lambda_\Sigma(s, x, u). \end{aligned} \tag{4.4}$$

Using convolution approximation on the test function and since  $\rho^{\text{FP}}$ ,  $\nabla_u \rho^{\text{FP}}$  and  $\gamma^+(\rho^{\text{FP}})$  are square-integrable, one can extend the preceding formula for all  $\Psi \in \mathcal{C}_b(\overline{Q_t}) \cap \mathcal{C}_b^{1,1,2}(Q_t)$ . Using the proof steps of Proposition 2.5, we know that  $(s, x, u) \in Q_t \mapsto \Gamma_n^\psi(t-s, x, u)$  is a smooth function that satisfies the equation:

$$\begin{cases} \partial_s \Gamma_n^\psi(t-s) + (u \cdot \nabla_x \Gamma_n^\psi(t-s)) + \frac{\sigma^2}{2} \Delta_u \Gamma_n^\psi(t-s) = 0 \text{ on } Q_t, \\ \lim_{s \rightarrow t} \Gamma_n^\psi(t-s, x, u) = \psi(x, u) \text{ on } \mathcal{D} \times \mathbb{R}^d, \\ \Gamma_n^\psi(t-s, x, u) = \Gamma_{n-1}^\psi(t-s, x, u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)) \text{ on } \Sigma_t^+. \end{cases} \tag{4.5}$$

Hence, in the case  $\Psi = \Gamma_n^\psi$ , (4.4) reduces to

$$\begin{aligned} &\int_{\mathcal{D} \times \mathbb{R}^d} \rho^{\text{FP}}(t, x, u) \psi(x, u) dx du \\ &= \int_{\mathcal{D} \times \mathbb{R}^d} \rho_0(x, u) \Gamma_n^\psi(t, x, u) dx du + \int_{Q_t} (B[x; \rho^{\text{FP}}(s)] \cdot \nabla_u \Gamma_n^\psi(t-s, x, u)) \rho^{\text{FP}}(s, x, u) ds dx du \\ &\quad - \int_{\Sigma_t^-} (u \cdot n_{\mathcal{D}}(x)) \gamma^-(\rho^{\text{FP}})(s, x, u) (\Gamma_n^\psi(t-s, x, u) - \Gamma_{n-1}^\psi(t-s, x, u)) d\lambda_\Sigma(s, x, u). \end{aligned} \tag{4.6}$$

Owing to (2.26), we obtain (4.3) by taking the limit  $n \rightarrow +\infty$ .

For (ii), let  $\mathbb{Q}$  be defined as in (4.1). Let us also introduce the time-marginal probability measures  $(\mu^m(t))$  related to the law of the stopped process  $(x(t \wedge \tau_m), u(t \wedge \tau_m))$  where  $\tau_m$  is the  $m^{\text{th}}$ -time  $x(t)$  hits  $\partial\mathcal{D}$ . Since  $(t, x) \mapsto B[x; \rho^{\text{FP}}(t)]$  is uniformly bounded, owing to Girsanov transform and Lemma 2.3, one can easily check that  $\mathbb{Q}$  is absolutely continuous w.r.t.  $\mathbb{P}$  so that the sequence  $\tau_m$  is well-defined and grows to  $\infty$  under  $\mathbb{Q}$ . As a first step, we show the existence of a  $L^\infty((0, T); L^2(\mathcal{D} \times \mathbb{R}^d))$ -density of  $\mu^m$ . Using a Riesz's representation argument, it is sufficient to show that there exists  $K > 0$ , possibly depending on  $M$ , such that

$$\forall \psi \in \mathcal{C}_c^\infty(\mathcal{D} \times \mathbb{R}^d), \text{ nonnegative, } \int_{\mathcal{D} \times \mathbb{R}^d} \psi(x, u) \mu^m(t, dx, du) \leq K \|\psi\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}. \quad (4.7)$$

To prove that (4.7) holds true, let us observe that for all nonnegative  $\psi \in \mathcal{C}_c^\infty(\mathcal{D} \times \mathbb{R}^d)$ , and for every  $\alpha \in (1, +\infty)$

$$\begin{aligned} \int_{\mathcal{D} \times \mathbb{R}^d} \psi(x, u) \mu^m(t, dx, du) &= \mathbb{E}_{\mathbb{Q}} [\psi(x(t \wedge \tau_m), u(t \wedge \tau_m))] \\ &\leq \exp \left\{ \frac{\|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)} T}{2(\alpha + 1)} \right\} (\mathbb{E}_{\mathbb{P}} [\psi^\alpha(x(t \wedge \tau_m), u(t \wedge \tau_m))])^{\frac{1}{\alpha}}, \end{aligned} \quad (4.8)$$

using Girsanov's change of probability and the boundedness of  $B^{\rho^{\text{FP}}(s)}$ . We will specify the appropriate  $\alpha$  later. Now observe that

$$\mathbb{E}_{\mathbb{P}} [\psi^\alpha(x(t \wedge \tau_m), u(t \wedge \tau_m))] = \langle \Gamma_m^{(\psi^\alpha)}(t), \mu_0 \rangle,$$

for  $\Gamma_m^\psi$  given as in (2.16). Let us observe that, according to Lemma 2.8, for all  $\beta \in (1, 2)$ , it holds that

$$\|\Gamma_m^{(\psi^\alpha)}(t)\|_{L^\beta(\mathcal{D} \times \mathbb{R}^d)}^\beta + \|\Gamma_m^{(\psi^\alpha)}\|_{L^\beta(\Sigma_t^-)}^\beta \leq \|\psi^\alpha\|_{L^\beta(\mathcal{D} \times \mathbb{R}^d)}^\beta + \|\Gamma_{m-1}^{(\psi^\alpha)}\|_{L^\beta(\Sigma_t^-)}^\beta.$$

Iterating this inequality  $M$  times and since  $\Gamma_1^{(\psi^\alpha)} = \psi = 0$  on  $\Sigma_T^+$ , one gets

$$\|\Gamma_m^{(\psi^\alpha)}(t)\|_{L^\beta(\mathcal{D} \times \mathbb{R}^d)}^\beta \leq M \|\psi^\alpha\|_{L^\beta(\mathcal{D} \times \mathbb{R}^d)}^\beta.$$

It thus follows that

$$\left| \langle \Gamma_m^{(\psi^\alpha)}(t), \mu_0 \rangle \right| \leq \|\Gamma_m^{(\psi^\alpha)}(t)\|_{L^\beta(\mathcal{D} \times \mathbb{R}^d)} \|\rho_0\|_{L^{\beta^*}(\mathcal{D} \times \mathbb{R}^d)} \leq m^{\frac{1}{\beta}} \|\psi^\alpha\|_{L^\beta(\mathcal{D} \times \mathbb{R}^d)}^\beta \|\rho_0\|_{L^{\beta^*}(\mathcal{D} \times \mathbb{R}^d)}^{\beta^*}. \quad (4.9)$$

Coming back to (4.8), we deduce that

$$\int_{\mathcal{D} \times \mathbb{R}^d} \psi(x, u) \mu^m(t, dx, du) \leq M^{\frac{1}{\beta\alpha}} \exp \left\{ \frac{\|b\|_{L^\infty} T}{2(\alpha + 1)} \right\} \|\psi\|_{L^{\beta\alpha}(\mathcal{D} \times \mathbb{R}^d)}^{\beta\alpha} (\|\rho_0\|_{L^{\beta^*}(\mathcal{D} \times \mathbb{R}^d)})^{\frac{1}{\alpha}} \quad (4.10)$$

For the special case where  $\alpha$  and  $\beta$  are such that  $\beta\alpha = 2$  and owing to  $(H_{\text{VFP}})$ -(iii), we get (4.7) for a constant  $K$  depending only on  $T$ ,  $\|\rho_0\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}$  and  $\|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}$ .

According to Proposition 2.5,  $\Gamma_n^\psi$  is a classical solution to (4.5) and by Itô's formula it follows that,  $\mathbb{Q}$ -a.s.,

$$\begin{aligned} \psi(x(t \wedge \tau_m), u(t \wedge \tau_m)) &= \Gamma_n^\psi(t, x(0 \wedge \tau_m), u(0 \wedge \tau_m)) + \int_0^{t \wedge \tau_m} \nabla_u \Gamma_n^\psi(t-s, x(s), u(s)) (dw(s) + B[x(s); \rho^{\text{FP}}(s)] ds) \\ &\quad + \sum_{k=0}^m (\Gamma_n^\psi(t - \tau_k, x(\tau_k), u(\tau_k)) - \Gamma_n^\psi(t - \tau_k, x(\tau_k), u(\tau_k^-))) 1_{\{\tau_m \leq t\}}. \end{aligned}$$

Since  $\Gamma_n^\psi(t - \tau_k, x(\tau_k), u(\tau_k^-)) = \Gamma_{n-1}^\psi(t - \tau_k, x(\tau_k), u(\tau_k))$ , taking the expectation on both sides of the equality yields

$$\begin{aligned} \langle \psi, \mu^m(t) \rangle &= \langle \Gamma_n^\psi(t), \mu_0 \rangle + \mathbb{E}_{\mathbb{Q}} \left[ \int_0^{t \wedge \tau_m} \nabla_u \Gamma_n^\psi(t-s, x(s), u(s)) B[x(s); \rho^{\text{FP}}(s)] ds \right] \\ &\quad + \mathbb{E}_{\mathbb{Q}} \left[ \sum_{k=0}^m (\Gamma_n^\psi(t - \tau_k, x(\tau_k), u(\tau_k)) - \Gamma_{n-1}^\psi(t - \tau_k, x(\tau_k), u(\tau_k))) 1_{\{\tau_k \leq t\}} \right]. \end{aligned} \quad (4.11)$$

We now take the limit of (4.11) as  $n$  tends to  $\infty$ . According to Lemma 2.4,  $\sum_{k=0}^m \mathbb{P}^\circ(\tau_k, x(\tau_k), u(\tau_k))^{-1}$  is a finite measure on  $\Sigma_T^-$  dominated by  $\lambda_{\Sigma_T}$ . By Girsanov, the same holds true for  $\sum_{k=0}^m \mathbb{Q}^\circ(\tau_k, x(\tau_k), u(\tau_k))^{-1}$ . Hence using the  $\lambda_\Sigma$ -a.e. convergence given in (2.26) we get that

$$\lim_{N \rightarrow +\infty} \mathbb{E}_\mathbb{Q} \left[ \sum_{k=0}^m \left( \Gamma_n^\psi(t - \tau_k, x(\tau_k), u(\tau_k)) - \Gamma_{n-1}^\psi(t - \tau_k, x(\tau_k), u(\tau_k)) \right) \right] = 0.$$

Since  $\Gamma_n^\psi$  and  $\nabla_u \Gamma_n^\psi$  converge to  $\Gamma^\psi$  and  $\nabla_u \Gamma^\psi$  in  $L^2(Q_T)$  and since  $\mu^m \in L^\infty((0, T); L^2(\mathcal{D} \times \mathbb{R}^d))$  we get

$$\langle \Gamma^\psi(t), \mu^m(t) \rangle = \langle \Gamma^\psi(t), \mu_0 \rangle + \mathbb{E}_\mathbb{Q} \left[ \int_0^{t \wedge \tau_m} \nabla_u \Gamma_n^\psi(t - s, x(s), u(s)) B[x(s); \rho^{\text{FP}}(s)] ds \right]$$

Next, one can observe that

$$\begin{aligned} \langle \psi, \mu^m(t) \rangle &= \int_{\mathcal{D} \times \mathbb{R}^d} \psi(x, u) \rho^m(t, x, u) dx du \\ &= \int_{\mathcal{D} \times \mathbb{R}^d} \Gamma^\psi(t)(x, u) \rho_0(x, u) dx du + \int_0^t (\nabla_u \Gamma^\psi(t, s), B[x; \rho^{\text{FP}}(s)]) \rho^m(s, x, u) ds dx du \quad (4.12) \\ &\leq \|\Gamma^\psi(t)\|_{L^2(\mathcal{D} \times \mathbb{R}^d)} \|\rho_0\|_{L^2(\mathcal{D} \times \mathbb{R}^d)} + \|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)} \|\nabla_u \Gamma^\psi\|_{L^2(Q_t)} \|\rho^m\|_{L^2(Q_t)}. \end{aligned}$$

Therefore, that  $\mu^m$  is uniformly bounded in  $L^2(Q_T)$  and since  $\rho^m$  tends to  $\rho$  a.e. on  $Q_T$ ,  $\rho^m$  converges weakly toward  $\rho$  in  $L^2(Q_T)$ . Taking the limit  $m \rightarrow +\infty$  in (4.12), we thus conclude on (ii).  $\square$

## 4.2 Uniqueness

Let us observe that, under (H), for any solution in law  $\mathbb{Q} \in \Pi_\omega$  to (1.1), its time marginal densities  $\rho(t) \in L^2(\omega, \mathcal{D} \times \mathbb{R}^d)$  are solution of the weighted nonlinear mild equation

$$\begin{aligned} \langle \sqrt{\omega} \psi, \rho(t) \rangle &= \langle \Gamma^\psi(t), \sqrt{\omega} \rho_0 \rangle + \int_0^t \langle \nabla_u \Gamma^\psi(t - s), \sqrt{\omega} B[\cdot; \rho(s)] \rho(s) \rangle ds \\ &\quad + \int_0^t \langle (\Gamma^\psi(t - s), (\nabla_u \log(\sqrt{\omega}) \cdot \sqrt{\omega} B[\cdot; \rho(s)]) \rho(s) \rangle ds + \frac{\sigma^2}{2} \int_0^t \langle (\Delta_u \sqrt{\omega}) \Gamma^\psi(t - s), \rho(s) \rangle ds. \end{aligned} \quad (4.13)$$

The proof of (4.13) simply replicate some proof steps of Theorem 4.2-(ii): for fixed  $t \in (0, T]$ ,  $m, n \in \mathbb{N} \setminus \{0\}$ , using Proposition 2.5, Itô's formula applied to  $(s, x, u) \in Q_t \mapsto \sqrt{\omega(u)} \Gamma_n^\psi(t - s, x, u)$  yields

$$\begin{aligned} &\mathbb{E}_\mathbb{Q} \left[ \sqrt{\omega(u(t))} \psi(x(t \wedge \tau_m), u(t \wedge \tau_m)) \right] \\ &= \mathbb{E}_\mathbb{Q} \left[ \sqrt{\omega(u(0))} \Gamma_n^\psi(t, x(0), u(0)) \right] + \mathbb{E}_\mathbb{Q} \left[ \int_0^{t \wedge \tau_m} \sqrt{\omega(u(s))} (\nabla_u \Gamma_n^\psi(t - s, x(s), u(s)) \cdot B[x(s); \rho(s)]) ds \right] \\ &\quad + \mathbb{E}_\mathbb{Q} \left[ \int_0^{t \wedge \tau_m} \Gamma_n^\psi(t - s, x(s), u(s)) \left( \left( \nabla_u \sqrt{\omega(u(s))} \cdot B[x(s); \rho(s)] + \frac{\sigma^2}{2} \Delta_u \sqrt{\omega(u(s))} \right) \right) ds \right] \\ &\quad + \mathbb{E}_\mathbb{Q} \left[ \sum_{k=0}^m \sqrt{\omega(u(\tau_k))} \left( \Gamma_n^\psi(t - \tau_k, x(\tau_k), u(\tau_k)) - \Gamma_{n-1}^\psi(t - \tau_k, x(\tau_k), u(\tau_k)) \right) 1_{\{\tau_k \leq t\}} \right]. \end{aligned}$$

Then taking the limit  $n \rightarrow +\infty$  and next  $m \rightarrow +\infty$ , the boundary term vanishes and the above gives (4.13). Since the drift coefficient in (1.1) is bounded, the fact that two solutions in law of (1.1) coincide is equivalent with the equality between the time marginal densities of these two solutions. We thus conclude the proof of the uniqueness part of Theorem 1.2 with

**Lemma 4.3.** *Under (H), any solution  $\rho(t) \in L^2(\omega; \mathcal{D} \times \mathbb{R}^d)$  to the non-linear mild equation (4.13) is equal to  $\rho^{\text{FP}}(t)$  for all  $t \in [0, T]$ .*

*Proof.* Using (4.13), for all  $t \in [0, T]$ , we have

$$\begin{aligned}
& \|\rho^{\text{FP}}(t) - \rho(t)\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 \\
&= \|\sqrt{\omega}(\rho^{\text{FP}}(t) - \rho(t))\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}^2 = \left( \sup_{\psi \in L^2(\mathcal{D} \times \mathbb{R}^d)} \langle \sqrt{\omega} \psi, \rho^{\text{FP}}(t) - \rho(t) \rangle \right)^2 \\
&\leq 2 \sup_{\substack{\psi \in L^2(\mathcal{D} \times \mathbb{R}^d); \\ \|\psi\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}=1}} \left( \int_0^t \langle \nabla_u \Gamma^\psi(t-s) + \Gamma(t-s) \nabla_u \log(\sqrt{\omega}), \sqrt{\omega} (B[\cdot; \rho^{\text{FP}}(s)] \rho^{\text{FP}}(s) - B[\cdot; \rho(s)] \rho(s)) \rangle ds \right)^2 \\
&\quad + \sigma^2 \sup_{\substack{\psi \in L^2(\mathcal{D} \times \mathbb{R}^d); \\ \|\psi\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}=1}} \left( \int_0^t \langle \Gamma(t-s) \Delta_u \sqrt{\omega}, \rho^{\text{FP}}(s) - \rho(s) \rangle ds \right)^2.
\end{aligned}$$

Using Cauchy–Schwartz’s inequality and Lemma 3.4, it follows that

$$\begin{aligned}
& \|\rho^{\text{FP}}(t) - \rho(t)\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 \\
&\leq 2 \sup_{\substack{\psi \in L^2(\mathcal{D} \times \mathbb{R}^d); \\ \|\psi\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}=1}} \left( \|\nabla_u \Gamma^\psi\|_{L^2(Q_t)}^2 + \frac{\alpha}{4} \|\Gamma^\psi\|_{L^2(Q_t)}^2 \right) \int_0^t \|\rho^{\text{FP}}(s) B[\cdot; \rho^{\text{FP}}(s)] - \rho(s) B[\cdot; \rho(s)]\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 ds \\
&\quad + \sigma^2 \left( 2\alpha \left( \frac{\alpha}{2} - 1 \right) + \alpha d \right) \left( \sup_{\substack{\psi \in L^2(\mathcal{D} \times \mathbb{R}^d); \\ \|\psi\|_{L^2(\mathcal{D} \times \mathbb{R}^d)}=1}} \|\Gamma^\psi\|_{L^2(Q_t)}^2 \right) \int_0^t \|\rho^{\text{FP}}(s) - \rho(s)\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 ds.
\end{aligned}$$

Using Corollary 2.7, one deduce that

$$\begin{aligned}
\|\rho^{\text{FP}}(t) - \rho(t)\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 &\leq \left( \frac{2}{\sigma^2} + \frac{\alpha T}{4} \right) \int_0^t \|\rho^{\text{FP}}(s) B[\cdot; \rho^{\text{FP}}(s)] - \rho(s) B[\cdot; \rho(s)]\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 ds \\
&\quad + \left( 2\alpha \left( \frac{\alpha}{2} - 1 \right) + \alpha d \right) \int_0^t \|\rho^{\text{FP}}(s) - \rho(s)\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 ds.
\end{aligned} \tag{4.14}$$

Now observe that

$$\begin{aligned}
& \int_{Q_T} \omega(u) (\rho^{\text{FP}}(s, x, u) B[x; \rho^{\text{FP}}(s)] - \rho(s, x, u) B[x; \rho(s)])^2 ds dx du \\
&\leq \|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)}^2 \|\rho^{\text{FP}} - \rho\|_{L^2(\omega, Q_T)}^2 + \int_{(0, T) \times \mathcal{D}} \left( \int_{\mathbb{R}^d} \omega(u) (\rho^{\text{FP}}(s, x, u))^2 du \right) (B[x; \rho^{\text{FP}}(s)] - B[x; \rho(s)])^2 ds dx
\end{aligned}$$

and that

$$\begin{aligned}
(B[x; \rho^{\text{FP}}(s)] - B[x; \rho(s)]) &= \frac{\int_{\mathbb{R}^d} b(v) (\rho^{\text{FP}}(s, x, v) - \rho(s, x, v)) dv}{\int_{\mathbb{R}^d} \rho^{\text{FP}}(s, x, v) dv} \\
&\quad + \frac{\int_{\mathbb{R}^d} b(v) \rho^{\text{FP}}(s, x, v) dv \int_{\mathbb{R}^d} (\rho^{\text{FP}}(s, x, v) - \rho(s, x, v)) dv}{\left( \int_{\mathbb{R}^d} \rho^{\text{FP}}(s, x, v) dv \right) \left( \int_{\mathbb{R}^d} \rho(s, x, v) dv \right)}.
\end{aligned}$$

Using the Maxwellian bounds  $\overline{M}$  and  $\underline{M}$  of  $\rho^{\text{FP}}$  given in (3.3), one has

$$\begin{aligned}
& \int_{\mathbb{R}^d} \omega(u) (\rho^{\text{FP}}(s, x, u))^2 du \left( \frac{\int_{\mathbb{R}^d} b(v) (\rho^{\text{FP}}(s, x, v) - \rho(s, x, v)) dv}{\int_{\mathbb{R}^d} \rho^{\text{FP}}(s, x, v) dv} \right)^2 \\
&\leq \frac{\int_{\mathbb{R}^d} \omega(u) |\overline{M}(s, u)|^2 du}{\left| \int_{\mathbb{R}^d} \underline{M}(s, u) du \right|^2} \left| \int_{\mathbb{R}^d} b(v) (\rho^{\text{FP}}(s, x, v) - \rho(s, x, v)) dv \right|^2 \\
&\leq 2 \|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)} \frac{\sup_{t \in (0, T)} \int_{\mathbb{R}^d} \omega(u) |\overline{M}(s, u)|^2 du}{\inf_{t \in (0, T)} \left| \int_{\mathbb{R}^d} \underline{M}(s, u) du \right|^2} \int_{\mathbb{R}^d} \omega^{-1}(v) dv \int_{\mathbb{R}^d} \omega(v) |\rho^{\text{FP}}(s, x, v) - \rho(s, x, v)|^2 dv,
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\mathbb{R}^d} \omega(u) (\rho^{\text{FP}}(s, x, u))^2 du \left( \frac{\int_{\mathbb{R}^d} b(v) \rho^{\text{FP}}(s, x, v) dv \int_{\mathbb{R}^d} (\rho^{\text{FP}}(s, x, v) - \rho(s, x, v)) dv}{\left( \int_{\mathbb{R}^d} \rho^{\text{FP}}(s, x, v) dv \right) \left( \int_{\mathbb{R}^d} \rho(s, x, v) dv \right)} \right)^2 \\
& \leq \|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)} \frac{\int_{\mathbb{R}^d} \omega(u) (\rho^{\text{FP}}(s, x, u))^2 du}{\left( \int_{\mathbb{R}^d} \rho^{\text{FP}}(s, x, v) dv \right)^2} \left( \int_{\mathbb{R}^d} b(v) (\rho^{\text{FP}}(s, x, v) - \rho(s, x, v)) dv \right)^2 \\
& \leq \|b\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^d)} \frac{\sup_{t \in (0, T)} \int_{\mathbb{R}^d} \omega(u) |\overline{M}(s, u)|^2 du}{\inf_{t \in (0, T)} \left| \int_{\mathbb{R}^d} \underline{M}(s, u) du \right|^2} \int_{\mathbb{R}^d} \omega(v) |\rho^{\text{FP}}(s, x, v) - \rho(s, x, v)|^2 dv,
\end{aligned}$$

Therefore, for some constant  $C > 0$ ,

$$\int_{(0, T) \times \mathcal{D}} \left( \int_{\mathbb{R}^d} \omega(u) (\rho^{\text{FP}}(s, x, u))^2 du \right) (B[x; \rho^{\text{FP}}(s)] - B[x; \rho(s)])^2 ds dx \leq 2C \|\rho^{\text{FP}} - \rho\|_{L^2(\omega, Q_t)}^2$$

Coming back to (4.14), we deduce

$$\begin{aligned}
& \|\rho^{\text{FP}}(t) - \rho(t)\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 \\
& \leq C \sup_{\psi \in L^2(\mathcal{D} \times \mathbb{R}^d); \|\psi\|_{L^2(\mathcal{D} \times \mathbb{R}^d)} = 1} \int_0^t \|\nabla_u \Gamma^\psi(t-s)\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 ds \int_0^t \|\rho^{\text{FP}}(s) - \rho(s)\|_{L^2(\omega, \mathcal{D} \times \mathbb{R}^d)}^2 ds.
\end{aligned}$$

where  $C$  depends only on  $d, \sigma$  and  $\alpha$ . Using Gronwall's lemma, this ends the proof.  $\square$

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## A Appendix

### A.1 Some recalls

**Corollary A.1.** ([Rana [27], p. 281]) *If  $\phi \in L^p(\mathbb{R}^d)$  for  $p \in [1, +\infty)$  then*

$$\lim_{|\delta| \rightarrow 0^+} \int |\phi(z + \delta) - \phi(z)|^p dz = 0.$$

**Theorem A.2** ([30], Chapter 4). *Let  $\mathcal{V}$  be an open subset of  $\mathbb{R}^d$  and  $\psi \in L^2(\mathcal{V})$  such that  $\nabla_v \psi \in L^2(\mathcal{V})$ . Then  $\nabla_v(\psi)^+, \nabla_v(\psi)^- \in L^2(\mathcal{V})$  with  $\partial_{v_i}(\psi)^+ = \partial_{v_i} \psi \mathbb{1}_{\{\psi \geq 0\}}$  and  $\partial_{v_i}(\psi)^- = -\partial_{v_i} \psi \mathbb{1}_{\{\psi \leq 0\}}$ .*

**Theorem A.3** (Lions [21]). *Let  $E$  be a Hilbert space with the inner product  $(\cdot, \cdot)_E$ . Let  $F \subset E$  equipped with the norm  $\|\cdot\|_F$  such that the canonical injection of  $F$  into  $E$  is continuous. Assume that  $A : E \times F \rightarrow \mathbb{R}$  is a bilinear application satisfying*

1.  $\forall \psi \in F$ , l'application  $A(\cdot, \psi) : E \rightarrow \mathbb{R}$  is continuous.
2.  $A$  is coercive; that is there exists a constant  $c > 0$  such that  $A(\psi, \psi) \geq c\|\psi\|_F^2$ ,  $\forall \psi \in F$ .

*Then for all linear application  $L : F \rightarrow \mathbb{R}$ , continuous on  $(F, \|\cdot\|_F)$ , there exists  $S \in E$  such that  $A(S, \psi) = L(\psi)$ ,  $\forall \psi \in F$ .*

### A.2 Proof of Lemma 3.7

In the case where  $\rho_0 \in L^2(\mathcal{D} \times \mathbb{R}^d)$ ,  $q \in L^2(\Sigma_T^-)$  and  $g \in L^2(Q_T)$  and where the drift coefficient  $B$  in equation (3.11) belongs to  $L^\infty((0, T) \times \mathcal{D})$ , Carrillo [9] established the existence and uniqueness of a weak solution  $S$  to (3.11) in  $\mathcal{H}(Q_T)$ , as well as the nonnegative property stated in Lemma 3.7. We slightly extend this result to the case where the solution space is  $\mathcal{H}(\omega, Q_T)$ . Since  $\mathcal{H}(\omega, Q_T) \subset \mathcal{H}(Q_T)$ ,

the uniqueness and the nonnegative property are obviously preserved. For the existence, let us replicate the original proof of [9].

Set

$$\begin{aligned}
E &= \mathcal{H}(\omega, Q_T), \\
F &= \{ \psi \in \mathcal{C}_c^\infty(\overline{Q_T}; \mathbb{R}) \text{ s.t. } \psi = 0 \text{ on } \{T\} \times \mathcal{D} \times \mathbb{R}^d \text{ and } \Sigma_T^+ \} \text{ equipped with the norm} \\
|\psi|_F^2 &= \int_{Q_T} \omega |\psi|^2 + \int_{\Sigma_T^-} |(u \cdot n_{\mathcal{D}})| \omega(u) |\psi(s, x, u)|^2 d\lambda_\Sigma(s, x, u) + \int_{Q_T} \omega |\nabla_u \psi|^2, \\
A(\phi, \psi) &= \int_{Q_T} \phi \mathcal{T}(\omega \psi) + \eta \int_{Q_T} \omega \psi \phi + \int_{Q_T} \omega \psi (B \cdot \nabla_u \phi) + \frac{\sigma^2}{2} \int_{Q_T} (\nabla_u(\omega \psi) \cdot \nabla_u \phi), \\
L(\psi) &= \int_{\mathcal{D} \times \mathbb{R}^d} \omega(u) \rho_0(x, u) \psi(0, x, u) dx du - \int_{\Sigma_T^-} (u \cdot n_{\mathcal{D}}(x)) \omega(u) \tilde{q}(t, x, u) \psi(t, x, u) d\lambda_\Sigma(t, x, u) - \int_{Q_T} \omega \tilde{g} \psi,
\end{aligned}$$

where  $\tilde{q}(t, x, u) = \exp\{-\eta t\} q(t, x, u)$ ,  $\tilde{g}(t, x, u) = \exp\{-\eta t\} g(t, x, u)$ , and  $\eta$  is a positive real parameter that we explicit later. Setting  $\|\psi\|_E := \sqrt{(\psi, \psi)_E}$  for the norm of  $E$ , we observe that  $\|\psi\|_E \leq |\psi|_F$  for  $\psi \in F$ . The continuity of the injection  $J : F \rightarrow E$  obviously hold true, as well as the continuity of the application  $A(\cdot, \psi) : E \rightarrow \mathbb{R}$  for  $\psi \in F$  fixed. In concern of the coercivity of  $A$ , we check that, for all  $\psi \in F$ ,

$$A(\psi, \psi) = \frac{1}{2} \|\psi(0)\|_{L^2(\omega, Q_T)}^2 + \|\psi\|_{L^2(\omega, \Sigma^-)}^2 + \frac{\sigma^2}{2} \|\nabla_u \psi\|_{L^2(\omega, Q_T)}^2 + \int_{Q_T} \left( \eta \omega - \frac{1}{2} (B \cdot \nabla_u \omega) - \frac{\sigma^2}{4} \Delta_u \omega \right) |\psi|^2.$$

According to Lemma 3.4, for all  $(t, x, u) \in Q_T$ ,

$$-\frac{1}{2} (B(t, x) \cdot \nabla_u \omega(u)) - \frac{\sigma^2}{4} \Delta_u \omega(u) \geq \left( -\frac{\alpha \|B\|_{L^\infty((0, T) \times \mathcal{D}; \mathbb{R}^d)}}{2} - \frac{\sigma^2}{4} \left( 2\alpha \left( \frac{\alpha}{2} - 1 \right) + \alpha d \right) \right) \omega(u).$$

Therefore, the coercivity of  $A$  on  $F$  is established by choosing  $\eta$  large enough. Theorem A.3 ensure the existence of  $\tilde{f} \in E$  such that, for all  $\phi$  in  $F$ ,

$$\begin{aligned}
& \int_{Q_T} \tilde{f} \mathcal{T}(\omega \phi) + \eta \int_{Q_T} \omega \tilde{f} \phi + \int_{Q_T} \omega \phi (B \cdot \nabla_u \tilde{f}) + \frac{\sigma^2}{2} \int_{Q_T} (\nabla_u(\omega \phi) \cdot \nabla_u \tilde{f}) + \int_{Q_T} \omega \tilde{g} \phi \\
&= \int_{\mathcal{D} \times \mathbb{R}^d} \omega(u) \phi(0, x, u) \rho_0(x, u) dx du - \int_{\Sigma_T^-} (u \cdot n_{\mathcal{D}}(x)) \omega(u) \tilde{q}(s, x, u) \phi(s, x, u) d\lambda_\Sigma(s, x, u).
\end{aligned}$$

Using the preceding expression with  $t = T$  and  $\phi = \frac{\psi}{\sqrt{\omega}}$  for  $\psi \in \mathcal{C}_c^\infty(Q_T)$ , we observe that

$$\|\mathcal{T}(\sqrt{\omega} \tilde{f})\|_{\mathcal{H}'(Q_T)} \leq \left( \eta + \|B\|_{L^\infty((0, T) \times \mathcal{D}; \mathbb{R}^d)} + \frac{\sigma^2}{2} \left( 1 + \frac{\alpha}{2} \right) \right) \|\tilde{f}\|_E + \|\tilde{g}\|_{L^2(\omega, Q_T)}.$$

According to Corollary 3.6,  $\tilde{f}$  also admits traces on the boundaries of  $Q_t$  satisfying the Green formula (3.7). In particular, for all  $\psi \in \mathcal{C}_c^\infty(\overline{Q_T})$  vanishing on  $\{T\} \times \mathcal{D} \times \mathbb{R}^d$  and on  $\Sigma_T^+$ ,

$$\begin{aligned}
& \left( \mathcal{T}(\tilde{f}), \psi \right)_{\mathcal{H}'(Q_T), \mathcal{H}(Q_T)} - \left( \mathcal{T}(\psi), \tilde{f} \right)_{\mathcal{H}'(Q_T), \mathcal{H}(Q_T)} \\
&= \int_{\mathcal{D} \times \mathbb{R}^d} \tilde{f}(0, x, u) \psi(0, x, u) dx du - \int_{\Sigma_T^-} (u \cdot n_{\mathcal{D}}(x)) \psi(s, x, u) \gamma^-(\tilde{f})(s, x, u) d\lambda_\Sigma(s, x, u).
\end{aligned}$$

From this expression, replicating the arguments of [9], we establish that  $\tilde{f}(0, \cdot) = \rho_0$  on  $\mathcal{D} \times \mathbb{R}^d$  and

$\gamma^-(\tilde{f}) = \tilde{q}$  on  $\Sigma_T^-$ . Hence, we obtain that

$$\begin{aligned} & \int_{Q_t} \tilde{f} \mathcal{T}(\psi) + \eta \int_{Q_t} \tilde{f} \psi + \int_{Q_t} \psi (B \cdot \nabla_u \tilde{f}) + \frac{\sigma^2}{2} \int_{Q_t} (\nabla_u \psi \cdot \nabla_u \tilde{f}) \\ &= - \int_{\mathcal{D} \times \mathbb{R}^d} \tilde{f}(t, x, u) \psi(t, x, u) dx du + \int_{\mathcal{D} \times \mathbb{R}^d} \rho_0(x, u) \psi(0, x, u) dx du \\ & \quad - \int_{\Sigma_t^+} (u \cdot n_{\mathcal{D}}(x)) \gamma^+(\tilde{f})(s, x, u) \psi(s, x, u) d\lambda_{\Sigma}(s, x, u) \\ & \quad - \int_{\Sigma_T^-} (u \cdot n_{\mathcal{D}}(x)) \tilde{g}(s, x, u) \psi(s, x, u) d\lambda_{\Sigma}(s, x, u), \end{aligned}$$

or equivalently that

$$\begin{cases} \mathcal{T}(\tilde{f}) + \eta \tilde{f} = (B \cdot \nabla_u \tilde{f}) + \frac{\sigma^2}{2} \Delta_u \tilde{f} \text{ on } \mathcal{H}'(Q_T), \\ \tilde{f}(0, x, u) = \rho_0(x, u) \text{ in } \mathcal{D} \times \mathbb{R}^d, \\ \gamma^-(\tilde{f})(t, x, u) = \tilde{g}(t, x, u) \text{ in } \Sigma_T^-. \end{cases}$$

Taking

$$f(t, x, u) = \exp\{\eta t\} \tilde{f}(t, x, u),$$

and

$$\|\mathcal{T}(\sqrt{\omega} f)\|_{\mathcal{H}'(Q_t)} \leq \left( \frac{\sigma^2}{2} \left(1 + \frac{\alpha}{2}\right) + \|B\|_{L^\infty((0,t) \times \mathcal{D})} \right) \|\nabla_u f\|_{L^2(\omega, Q_t)} + \|g\|_{L^2(\omega, Q_t)} < +\infty. \quad (\text{A.2})$$

we deduce that  $f \in \mathcal{H}(\omega, Q_T)$  is a weak solution to (3.11).

*Proof of (i1).* Since  $2\mu > 1$ , by Jensen's inequality, it holds  $m^{2\mu}(t, u) \leq G(\sigma^2 t, \cdot) * p_0^2(u)$ . Setting  $f(u) := (1 + |u|)\omega(u)$ , we thus have, by Lemma 3.4 (i),

$$\begin{aligned} \int_{\mathbb{R}^d} (1 + |u|)\omega(u) |p(t, u)|^2 du &\leq \exp\{2at\} \int_{\mathbb{R}^d} f(u) G(\sigma^2 t) * |p_0|^2(u) du \\ &\leq 2^{\frac{\alpha}{2}} \exp\{at\} \left( \|p_0\|_{L^2(\mathbb{R}^d)}^2 \int_{\mathbb{R}^d} f(u) G(\sigma^2 t, u) du + \int_{\mathbb{R}^d} f(u') |p_0(u')|^2 du' \right). \end{aligned}$$

Since the Gaussian function has all its moments finite,  $\int_{\mathbb{R}^d} f(u) G(\sigma^2 t, u) du < +\infty$  and (i1) follows from the assumption (3.13).

*Proof of (i2).* Let us remark that

$$\int_{\mathbb{R}^d} p(t, u) du \geq \exp\{-|a|T\} \int_{\mathbb{R}^d} \left( G(\sigma^2 t) * p_0^{\frac{1}{\mu}}(u) \right)^\mu du.$$

In addition, since  $\mu > 1$ , Hölder's inequality yields

$$\left( \int_{\mathbb{R}^d} G(\nu, u) \left( G(\sigma^2 t, \cdot) * p_0^{\frac{1}{\mu}}(u) \right) du \right)^\mu \leq \|G(\sigma^2 \nu)\|_{L^{\mu'}(\mathbb{R}^d)}^\mu \int_{\mathbb{R}^d} \left( G(\sigma^2 t) * p_0^{\frac{1}{\mu}}(u) \right)^\mu du,$$

where  $\mu'$  is the conjugate of  $\mu$ , and  $\nu$  is a positive constant and with  $\|G(\nu)\|_{L^{\mu'}(\mathbb{R}^d)}^\mu = (2\pi\sigma^2\nu)^{-\frac{d}{2}} (\mu')^{\frac{-d}{2(\mu'-1)}} > 0$ . Setting  $C_{\mu, \nu} := \exp\{|a|T\} \|G(\sigma^2 \nu)\|_{L^{\mu'}(\mathbb{R}^d)}^\mu$ , we then observe that

$$\begin{aligned} \inf_{t \in (0, T)} \int_{\mathbb{R}^d} p(t, u) du &\geq C_{\mu, \nu}^{-1} \inf_{t \in (0, T)} \left( \int_{\mathbb{R}^d} G(\sigma^2 \nu, u) G(\sigma^2 t) * p_0^{\frac{1}{\mu}}(u) du \right)^\mu \\ &\geq C_{\mu, \nu}^{-1} \inf_{t \in (0, T)} \left( \int_{\mathbb{R}^d} G(\sigma^2 \nu) * G(t, u) p_0^{\frac{1}{\mu}}(u_0) du_0 \right)^\mu \\ &\geq C_{\mu, \nu}^{-1} \left( \frac{1}{2\pi\sigma^2(\nu + t)} \right)^{\frac{d\mu}{2}} \left( \int_{\mathbb{R}^d} \exp\left\{ \frac{-|u_0|^2}{2\sigma^2(\nu + t)} \right\} p_0^{\frac{1}{\mu}}(u_0) du_0 \right)^\mu, \end{aligned}$$

by using the properties of convolution product between Gaussian functions. We thus get the lower-bound

$$\inf_{t \in (0, T)} \int_{\mathbb{R}^d} p(t, u) du \geq C_{\mu, \nu}^{-1} \left( \frac{1}{2\pi\sigma^2(T + \nu)} \right)^{\frac{d\mu}{2}} \left( \int_{\mathbb{R}^d} \exp \left\{ \frac{-|u_0|^2}{2\sigma^2\nu} \right\} p_0^{\frac{1}{\mu}}(u_0) du_0 \right)^{\mu}.$$

Since  $p_0$  is supposed to be not identically zero on  $\mathbb{R}^d$ , we conclude (i2).

Proof of (i3). For all  $\mu > 0$ , it is enough to show that for some real sequence  $\{\epsilon_k; k \in \mathbb{N}\}$  decreasing to 0; it holds

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^d} \left| p^{\frac{1}{\mu}}(\epsilon_k, u) - p_0^{\frac{1}{\mu}}(u) \right|^{2\mu} du = 0. \quad (\text{A.3})$$

Indeed, in the case  $\mu \leq 1$ , recalling that:

$$||c| - |b||^q \leq ||c|^q - |b|^q|, \text{ for } c, b \in \mathbb{R}, q \geq 1,$$

it holds, for all  $k$ ,

$$|p(\epsilon_k, u) - p_0(u)|^2 = \left( |p(\epsilon_k, u) - p_0(u)|^{\frac{1}{\mu}} \right)^{2\mu} \leq |p^{\frac{1}{\mu}}(\epsilon_k, u) - p_0^{\frac{1}{\mu}}(u)|^{2\mu}.$$

Then (A.3) implies (i3) for all  $\{\epsilon_k; k \in \mathbb{N}\}$  considered above

In the case  $\mu > 1$ , (A.3) yields the convergence  $p(\epsilon_k) \rightarrow p_0(\cdot)$  in  $L^{2\mu}(\mathbb{R}^d)$ . Applying [Theorem 4.9, Brézis [7]], we deduce the existence of a subsequence of  $\{p(\epsilon_k); k \in \mathbb{N}\}$  such that

$$\lim_{k \rightarrow +\infty} p(\epsilon_k, u) = p_0(u) \text{ a.e. } u \in \mathbb{R}^d, \text{ and } \sup_{k \in \mathbb{N}} |p(\epsilon_k, u)|^{\frac{1}{\mu}} \in L^{2\mu}(\mathbb{R}^d).$$

Since  $\sup_{k \in \mathbb{N}} |p(\epsilon_k, u)|^2 \leq (\sup_{k \in \mathbb{N}} |p(\epsilon_k, u)|^{\frac{1}{\mu}})^{2\mu}$ , (i3) follows from the Lebesgue Dominated Convergence Theorem.

We now check that (A.3) holds true: By definition  $\int_{\mathbb{R}^d} |p^{\frac{1}{\mu}}(t, u) - p_0^{\frac{1}{\mu}}(u)|^{2\mu} du$  is bounded from above by

$$|\exp \{2a\epsilon_k\} - 1| \int_{\mathbb{R}^d} |p_0(u)|^2 du + \exp \{2a\epsilon_k\} \int_{\mathbb{R}^d} |m(\epsilon_k, u) - p_0^{\frac{1}{\mu}}(u)|^{2\mu} du.$$

According to (3.13)  $\int_{\mathbb{R}^d} |p_0(u)|^2 du$  is finite so the first term in the expression tends to 0 when  $k$  goes to infinity. For the second term, a change of variables and Hölder's inequality give

$$\begin{aligned} \int_{\mathbb{R}^d} |m(\epsilon_k, u) - p_0^{\frac{1}{\mu}}(u)|^{2\mu} du &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} G(\sigma^2, u_0) p_0^{\frac{1}{\mu}}(u - \sqrt{\epsilon_k} u_0) du_0 - p_0^{\frac{1}{\mu}}(u) \right|^{2\mu} du \\ &\leq \int_{\mathbb{R}^d} G(\sigma^2, u_0) \left( \int_{\mathbb{R}^d} |p_0^{\frac{1}{\mu}}(u - \sqrt{\epsilon_k} u_0) - p_0^{\frac{1}{\mu}}(u)|^{2\mu} du \right) du_0. \end{aligned}$$

Since  $p_0^{\frac{1}{\mu}} \in L^{2\mu}(\mathbb{R}^d)$ , the convergence result in Corollary (A.1) implies that

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^d} |p_0^{\frac{1}{\mu}}(u - \sqrt{\epsilon_k} u_0) - p_0^{\frac{1}{\mu}}(u)|^{2\mu} du = 0.$$

We check that

$$\int_{\mathbb{R}^d} G(\sigma^2, u_0) \sup_k \left( \int_{\mathbb{R}^d} |p_0^{\frac{1}{\mu}}(u - \sqrt{\epsilon_k} u_0) - p_0^{\frac{1}{\mu}}(u)|^{2\mu} du \right) du_0 \leq 2 \int_{\mathbb{R}^d} |p_0(u)|^2 du < +\infty,$$

in order to conclude that  $\lim_{t \rightarrow 0^+} \int_{\mathbb{R}^d} |m(t, u) - p_0^{\frac{1}{\mu}}(u)|^{2\mu} du = 0$  by dominated convergence.

Proof of (i4). For all  $\delta > 0$ ,  $(t, u) \in [\delta, T] \times \mathbb{R}^d \mapsto p(t, u)$  is  $C^\infty$  and

$$\partial_t p(t, u) = ap(t, u) + \mu \exp\{at\} \partial_t m(t, u) m^{\mu-1}(t, u),$$

from which we get, for all  $t > 0$ ,

$$\int_{\mathbb{R}^d} |\partial_t p(t, u)|^2 du \leq 2|a| \int_{\mathbb{R}^d} |p(t, u)|^2 du + 2\mu^2 \exp\{2at\} \int_{\mathbb{R}^d} |\partial_t m(t, u)|^2 m^{2(\mu-1)}(t, u) du.$$

By (i1),  $\int_{\mathbb{R}^d} |p(t, u)|^2 du$  is finite. For the second term, observe that

$$|\partial_t m(t, u)| \leq \int_{\mathbb{R}^d} \left| \frac{d}{dt} G(\sigma^2 t, u - u_0) \right| p_0^{\frac{1}{\mu}}(u_0) du_0,$$

where

$$\frac{d}{dt} G(\sigma^2 t, u - u_0) = \left( \frac{1}{2\pi\sigma^2 t} \right)^{\frac{d}{2}} \left( \frac{|u - u_0|^2 - \sigma^2 t}{2\sigma^2 t^2} \right) \exp \left\{ -\frac{|u - u_0|^2}{2\sigma^2 t} \right\}$$

Using the inequality

$$|z|^p \exp \left\{ \frac{-|z|^2}{4\theta} \right\} \leq (2p\theta)^{\frac{p}{2}} \text{ for } \theta > 0, p \geq 1, z \in \mathbb{R}, \quad (\text{A.4})$$

it follows that

$$|\partial_t m(t, u)| \leq \frac{d+1}{2\delta} \int_{\mathbb{R}^d} \left( \frac{1}{2\pi\sigma^2 t} \right)^{\frac{d}{2}} \exp \left\{ \frac{-|u - u_0|^2}{4\sigma^2 t} \right\} p_0^{\frac{1}{\mu}}(u_0) du_0 = \frac{2^{\frac{d}{2}}(d+1)}{2\delta} G(2\sigma^2 t) * p_0^{\frac{1}{\mu}}(u).$$

Using the upper-bound  $m(t, u) \leq 2^{\frac{d}{2}} G(2\sigma^2 t) * p_0^{\frac{1}{\mu}}(u)$ , it follows that

$$\begin{aligned} & \int_{(\delta, T) \times \mathbb{R}^d} \exp\{2at\} |\partial_t m(t, u)|^2 m^{2(\mu-1)}(t, u) dt du \\ & \leq 2^{d\mu} \left( \frac{d+1}{2\delta} \right)^{2\mu} \int_{(\delta, T) \times \mathbb{R}^d} \exp\{2at\} \left( G(2\sigma^2 t) * p_0^{\frac{1}{\mu}}(u) \right)^{2\mu} dt du \\ & \leq 2^{d\mu} \left( \frac{d+1}{2\delta} \right)^{2\mu} T \int_{\mathbb{R}^d} |p_0(u_0)|^2 du_0 \int_0^T \exp\{2at\} dt. \end{aligned}$$

Since  $p_0 \in L^2(\mathbb{R}^d)$ , we deduce (i4).

Proof of (i5). For the assertion  $\nabla_u p \in L^2((0, T) \times \mathbb{R}^d)$ , we use the sequence  $\{\epsilon_k; k \in \mathbb{N}\}$  given in (i3). For all  $k$ , it holds

$$\begin{aligned} \int_{(\epsilon_k, T) \times \mathbb{R}^d} |\nabla_u p(t, u)|^2 dt du &= \mu^2 \int_{(\epsilon_k, T) \times \mathbb{R}^d} \exp\{2at\} |\nabla_u m(t, u)|^2 m^{2\mu-2}(t, u) dt du \\ &= \frac{1}{2\mu-1} \int_{(\epsilon_k, T) \times \mathbb{R}^d} \exp\{2at\} (\nabla_u m(t, u) \cdot \nabla_u (m^{2\mu-1}(t, u))) dt du. \end{aligned} \quad (\text{A.5})$$

Since  $2\mu-1 > 0$  and  $|\nabla_u m(t, u)| \leq (\sigma^2 t)^{-1} \int_{\mathbb{R}^d} |u - u_0| G(t, u - u_0) p_0^{\frac{1}{\mu}}(u_0) du_0$ , the smoothness of  $G$  and the assumptions (3.13) yield

$$\lim_{|u| \rightarrow +\infty} |\nabla_u m(t, u)| m^{2\mu-1}(t, u) = 0, \text{ for all } t \in [\epsilon_k, T].$$

By integrating by parts the right member of (A.5) and using the heat equation  $\Delta_u m = \frac{2}{\sigma^2} \partial_t m$ , we get

$$\begin{aligned}
& \int_{(\epsilon_k, T) \times \mathbb{R}^d} \exp\{2at\} |\nabla_u m(t, u)|^2 m^{2\mu-2}(t, u) dt du \\
&= \frac{-1}{2\mu-1} \int_{(\epsilon_k, T) \times \mathbb{R}^d} \exp\{2at\} \Delta_u m(t, u) m^{2\mu-1}(t, u) dt du \\
&= \frac{-2}{\sigma^2(2\mu-1)} \int_{(\epsilon_k, T) \times \mathbb{R}^d} \exp\{2at\} \partial_t m(t, u) m^{2\mu-1}(t, u) dt du \\
&= \frac{-1}{\sigma^2 \mu(2\mu-1)} \int_{(\epsilon_k, T) \times \mathbb{R}^d} \exp\{2at\} \partial_t (m^{2\mu}(t, u)) dt du
\end{aligned}$$

Using again a integration by parts enable us to obtain the equality

$$\begin{aligned}
& \int_{(\epsilon_k, T) \times \mathbb{R}^d} \exp\{2at\} |\nabla_u m(t, u)|^2 m^{2\mu-2}(t, u) dt du \\
&= \frac{-1}{\sigma^2 \mu(2\mu-1)} \left( \int_{\mathbb{R}^d} |p(T, u)|^2 du - \int_{\mathbb{R}^d} |p(\epsilon_k, u)|^2 du \right) + \frac{2a}{\sigma^2 \mu(2\mu-1)} \int_{(\epsilon_k, T) \times \mathbb{R}^d} |p(t, u)|^2 dt du.
\end{aligned}$$

Coming back to (A.5) and letting  $k$  increase to  $+\infty$ , it follows that

$$\begin{aligned}
\int_{(0, T) \times \mathbb{R}^d} |\nabla_u p(t, u)|^2 dt du &= \frac{-\mu}{\sigma^2(2\mu-1)} \left( \int_{\mathbb{R}^d} |p(T, u)|^2 du - \int_{\mathbb{R}^d} |p_0(u)|^2 du \right) \\
&\quad + \frac{2a\mu}{\sigma^2(2\mu-1)} \int_{(0, T) \times \mathbb{R}^d} |p(t, u)|^2 dt du.
\end{aligned}$$

Thanks to the assumption (3.13),  $p_0 \in L^2(\mathbb{R}^d)$  and (i1), the right member is finite. We conclude (i5).